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INDIVIDUALIZED INSTRUCTION IN GRADE SEVEN MATHEMATICS: RATIONALE, DESCRIPTION, AND FEASIBILITY REPORT

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Individualized Instruction in Grade Seven Mathematics: Rationale, Description, and Feasibility Report" submitted by Marvin Lawton Westrom in partial fulfilment of the requirements for the degree of Master of Education.

ABSTRACT

This thesis is one of three produced as a direct result of the Hardisty Project. It presents a rationale of the innovative techniques used in individualizing instruction at Hardisty school, a comprehensive description of the methods and materials employed in the experiment, and an assessment of the methods and the effects of the program on students at a feasibility level.

An historical rationale traces the trends and development of individualized instruction beginning with the Dalton and Winnetka plans of the 1920's through computer assisted instruction. The theoretical rationale sets a basis for the devised approach to individualization and discusses the use of ability grouping and behavioral objectives in this context.

Four types of programmed learning packages were prepared for the experiment. They were designed so that students could proceed with a minimum of intervention by the teacher. Each topic was composed of two parts called Phase 1 and Phase 2. The Phase 1 packages contained a number of sections. Each section contained a list of behavioral objectives, a series of explanations, and a set of exercises. Special 'check-exercises' were used by students to pre-test themselves. Those who answered a check-exercise correctly could skip the section to which that exercise was keyed.

The Phase 2 packets were of three types. A test ad-

ministered after students completed their Phase 1 packets streamed them into three corresponding groups: Basic, Intermediate, and Advanced. Basic students worked on unachieved objectives which were deemed imperative to the course, Intermediate students continued in an attempt to master all objectives for the topic, and Advanced students were given new objectives related to the subjects of the topic. When students finished this Phase, they worked on problem solving and enrichment until the start of the next topic.

Students were allowed to attend a regular class if they found the self-study too difficult. A close correspondence was maintained between the exercises, check-exercises, test questions and the objectives. Mastery was expected for achievement of objectives, so that no errors were allowed.

The experiment showed that the devised procedures constitute a reasonable alternative to existing methods of instruction. Apparent drawbacks are the effort required to produce the materials and the amount of clerical work involved in using the method. Suggested improvements include the addition of materials designed to meet more specific needs of certain groups and the improvement of enrichment facilities.

Compared to traditional instruction, students learned more and accepted more responsibility for their learning under the experimental method. Both students and teachers indicated that they preferred the experimental method over regular instruction for mathematics.

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CHAPTER ONE

INTRODUCTION

1. ORIGINS OF THE HARDISTY EXPERIMENT

During the 1968-69 school year, an individualized mathematics program called I.P.I. (Individually Prescribed Instruction) was installed in Forest Heights Elementary school. Many of the grade six students who worked under this method enroled the next year in Hardisty Junior High school. In an effort to meet the needs of this large group of students, the teachers at Hardisty school began planning an individualized instruction program for grade seven. L. Throndson, the coordinator for that area contacted the University of Alberta for assistance.

During the school year 1967-68, R. Mortlock and J. Bierden were involved in an individualization project in Pontiac, Michigan. In 1969, Mortlock joined the staff of the University of Alberta, and subsequently became the contact for Throndson and the teachers at Hardisty school.

In the fall of 1969, the teachers at Hardisty installed an individualized program of their own devising in their mathematics classes. Meanwhile, Mortlock, assisted by S. Sigurdson and three graduate students developed a new program based on the ideas he had developed in Michigan. In

January of 1970, this new program was introduced to Hardisty school. The 'Hardisty Project', as this experiment came to be known, lasted from January through April of that year. It involved five teachers (including Throndson), nearly three hundred grade seven mathematics students, and is comprehensively reported in three masters theses: Sunde (1970), te Kampe (1970), and the present study.

2. AN OVERVIEW OF THE PROGRAM

Mortlock and his team (hereafter called the experimenters) developed an instructional model which called for packets of specially prepared typewritten pages (called 'materials'). Individualization was accomplished in four main ways:

1. The materials were 'programmed' so that students could proceed through them in different ways.

2. Students were streamed into three ability groups by virtue of previous achievement.

3. Students were given a choice of mode of instruction. They could proceed at their own pace using the materials, or optionally join a more traditional type classroom (called the 'teacher-taught class') at any time.

4. Teachers (except for the one handling the teacher-taught class), were relieved of lecturing duties to spend their time working with individuals or small groups of

students.

Four topics were covered during the experiment:

1. Rational Numbers and Fractions
2. Operations with Rational Numbers
3. Decimals
4. Ratio, Proportion and Per Cent

All students covered each topic by initially working through the concepts at the middle achievement level. They were then tested, and the results used to form three groups. The two lower groups (called Basic and Intermediate) reviewed the topic by using additional materials. The Basic group was not responsible however, for mastering all of the concepts covered. The top group (called Advanced) were also given a new set of materials, but these were directed towards new ideas rather than a review. When a student had finished his review or Advanced work, he wrote a second test on the topic. After this, he spent one or two periods working on Non-routine Problems, and was then allowed to work in an Enrichment Center until the start of the next topic.

The materials for each topic were arranged into sections. Each section contained a list of behavioral objectives for that section, a development or discussion of the ideas to be learned, and a set of exercises. If a student felt from reading the objectives that he already understood the ideas of that section, he could skip to the exercises. The first exercise in each set was called a 'check-exercise'.

The purpose of these special exercises was to allow the student to pre-test himself. After completing one of these, the student evaluated his own answer by checking with an answer sheet. If he had answered the question correctly, he could go on to the next section. Otherwise, he was to work through a number of further exercises.

Students worked at their own rates, and moved freely between classrooms to locate reference books, view filmstrips, seek help from a teacher, or attend special classes. The teachers' duties centered around helping individual students, and on the whole, a very informal atmosphere prevailed.

3. THE PURPOSE OF THIS STUDY

The purpose of this study was to provide a comprehensive overview of the Hardisty experiment. This report contains descriptions of both the planning and operation of the experiment, a feasibility assessment of the procedures employed and suggestions for improvement of the designed model for individualized instruction.

Fundamental to the planning of the experiment was both the historical and theoretical background of individualized instruction. The more relevant portions of these form a main part of this study: the rationale. A report of the events preparatory to and during the experiment form the

second main part: the description of the experiment. The third and final segment supplies a formative evaluation which consists of a feasibility report, and suggestions for improvement and further study.

4. THE NEED FOR THIS STUDY

It is the duty of any experimenter to collect and report as much useful information resulting from an experiment as can be obtained. Each of the three main parts of this study present information useful to the educator.

The rationale collects together the ideas that were used to generate the devised procedures. These ideas are an aide in understanding the Hardisty experiment. They are also useful in their own right as a collection of ideas felt by one group to be important when considering an individualized program.

The description of the experimental procedures records the results of a translation from the theory presented in the rationale to a practical application in the school. It describes numerous old and new educational ideas and shows how these can be organized into a consistent program for individualizing instruction.

The feasibility report provides the opportunity for a global evaluation of the devised program. From the reported reactions and outcomes, it would be possible for the

reader to consider the effectiveness of specific parts of the program, and the program as a whole.

In this manner, the study has resulted in the present report which provides information at a feasibility level. This information will be useful not only to aid in replication of the experiment, but also to help improve the teaching procedures devised and provide information at a practical classroom level to any teacher who might be contemplating similar individualization. The study is carried out on the basic premise that such practical classroom data is as valuable to the teaching profession as is statistical data. For other (more formal) evaluations of the present experiment, see Sunde (1970), and te Kampe (1970).

5. DELIMITATIONS

The delimitations of this study are as follows:

1. The rationale is concerned with the events and theories which had a direct bearing on the development of the procedures for the Hardisty experiment. The emphasis is on instruction management techniques, and only those techniques which differ significantly from normal teaching practice are rationalized.

2. The feasibility of any designed program depends upon the situation to which it is applied. The value of a feasibility report then, is in allowing a reader to extrapolate

the relevant outcomes and generalize them to his particular situation. Considering this, the feasibility report is limited to recorded observations of reactions of participants in the experiment. This procedure has the further advantage that evaluative value judgements made by the writer are capable of scrutiny by the reader.

6. DEFINITIONS

General Terms

CURRICULUM - the set of experiences provided by a school for its students which are directed towards educational goals.

VARIABLE CURRICULUM - a curriculum which intentionally provides different experiences for different students on the basis of some established criteria.

TRADITIONAL INSTRUCTION - instruction consisting of a combination of class presentation and seat-work which is directed towards the whole class rather than individuals or groups within the class.

INDIVIDUALIZED INSTRUCTION - instruction which is directed towards groups within a class or towards individual students.

SELF-STUDY - study which, although circumscribed by external restrictions, allows for individual determination of content and sequence by the student.

SELF-STUDY MATERIALS - workbooks, audio-visual material, or any other equipment available for students to use for self-study. The term 'materials' is used to mean specifically the sets of typewritten pages (workbooks) which the students received.

Technical Terms

ORIENTATION - the activity aimed at making the students aware of what was expected of them and the nature of the materials.

PHASE 1 MATERIAL - the first workbook used for each topic. These contained a flowchart, a record page, from ten to twelve sections, a review section, and a set of answers (in that order), directed at the intermediate level.

FLOWCHART - a single sheet of paper on which the possible sequences of study for a given topic had been diagrammed.

RECORD PAGE - a single sheet of paper on which teachers indicated which objectives a student had achieved for Phase 1, which he had to do for Phase 2, his group designation, and results of his problem solving activities.

SECTION - a number of pages consisting of a list of objectives, a development, and a set of activities and exercises.

REVIEW SECTION - the last section of each chapter which contained exercises for purposes of reviewing a topic, a vocabulary list, and a list of key ideas of the topic.

OBJECTIVE - (1) a statement of educational intent. (2) a statement of intent, a sample question which could be used to test the achievement of that intent, and a complete solution to that question indicating the criterion for marking.

DEVELOPMENT - a series of explanations, diagrams, examples, and questions designed to teach students the ideas required for a given section.

ACTIVITIES AND EXERCISES - a set of check-exercises, exercises, and directions for activities related to the objectives of a section.

CHECK-EXERCISE - a question which was parallel in difficulty and design to the sample test question given in the corresponding objectives of a section, and to the actual test questions which would be used to test achievement of the objectives.

TEST 1 - the test which students wrote after completion of their Phase 1 materials.

PHASE 1 - the activities associated with working through the Phase 1 materials and writing Test 1.

PHASE 2 MATERIAL - the second workbook used for each topic. There were three distinct sets called Basic, Intermediate, and Advanced Phase 2 material. The Basic set contained a second flowchart, directions to the student, prominent key ideas, check exercises, and exercises. The Intermediate set contained only exercises and check-exercises.

Both Basic and Intermediate Phase 2 materials were intended for use in reviewing the concepts covered in the Phase 1 material. The Advanced set contained a new set of objectives, a development, and a set of activities and exercises aimed at these new objectives.

TEST 2 - the test which students wrote after completion of their Phase 2 materials.

PHASE 2 - the activities associated with working through the Phase 2 materials and Test 2.

NON-ROUTINE PROBLEM SOLVING - a self-study activity externally bounded by a set of Non-Routine Problem Solving materials. These materials contained from fifteen to twenty problems which demanded solutions which were not specifically taught.

ENRICHMENT - a self-study activity directed by materials available in the 'Enrichment Center'. These included a problem box, directions for project work, mathematical games, directions for independent study, and other activities which the teachers and students determined.

TOPIC - the activities associated with working through Phase 1, Phase 2, the Non-Routine Problem Solving, and Enrichment.

7. OUTLINE OF THE REPORT

The purpose of this study was to provide a comprehensive overview of the Hardisty experiment. The study consists of three main parts: a rationale, a description, and a feasibility report.

The rationale is concerned with the background of the methods and procedures employed in the experiment. The second chapter of this study presents some of the events of the twentieth century which have led to the need for individualized instruction that exists today. Chapter III outlines the theoretical ideas used by the experimenters in devising and justifying the program that was implemented in Hardisty school.

The description of the experimental procedures is contained in Chapter IV. This chapter presents all of the procedures devised for the experiment, and explains them in some detail.

The feasibility report is the topic of Chapter V. Here, the reactions of the students, teachers, and experimenters have been summarized, and the observable outcomes of the experiment listed. Chapter VI concludes the study with an evaluation of the experiment and the theory behind the experimental methods, and a general statement on the role of individualized instruction in secondary schools.

CHAPTER TWO

AN HISTORICAL RATIONALE

It has long been recognized that attention to individual differences of the learner can improve the education process. Projects have been carried out in the past which can, by their successes or failures, provide insight into the problems involved in trying to account for these differences. The purpose of this chapter is to explore innovations which revealed concepts important to the current experiment.

The Dalton and Winnetka plans showed the trends of dissatisfaction with the school system in and after 1915; both of these plans were influenced by and had effects on European education at that time. It was at this same point in time that Pressey's work with objective testing led to the development of the teaching machine. Skinner and Crowder were the main proponents of two divergent streams of thought on the construction of teaching machines. With the development of the high-speed digital computer and the advent of time-sharing, the teaching machine concept has been extended to include computer assisted instruction and computer assisted instruction management. All of the above will be discussed in the present chapter along with some of the more relevant current projects in individualized instruction. These ideas are considered to form a part of the basis for decisions that were

made in devising the program used in the current experiment.

1. THE DALTON AND WINNETKA PLANS

The Dalton and Winnetka plans were both implemented shortly after 1910; both flourished under the nurture of one person and both seemed to loose their impact shortly after that person left. They were aimed at overcoming two dissimilar disagreeable aspects of the then current educational system, and in practice have entirely dissimilar appearances.

Miss Helen Parkhurst (with inspiration from Montessori) had responsibility for the Dalton Laboratory Plan. She reasoned that the school timetable made unreasonable demands on the students; a person's interests cannot change at the sound of a school class bell. A desire for knowledge about a certain subject, carefully instilled during a class period, is doomed to frustration as the student enters another teacher's class. Parkhurst also felt that this imposition did nothing valuable for the student, where allowing him to plan his own time and activities would enhance his ability to accept responsibility and make him a more useful member of society on graduation. This teaching method was first applied by Parkhurst on crippled children, but took its name from Dalton High school where it was first implemented with normal children by Mrs. W. Crane.

Generally, under this plan, students are given an

outline for a years work at the beginning of the school year.

> This (gives) him a perspective of the plan of his education. He will thus be able to judge of the steps he must take each month and each week so that he may cover the whole road, instead of going blindly forward with no idea of either the road or the goal. ...
>
> What does a pupil do when given ... responsibility for the performance of such and such work? Instinctively he seeks the best way of achieving it. Then having decided, he proceeds to act upon that decision. Supposing his plan does not seem to fit his purpose, he discards it and tries another. Later on he may find it profitable to consult his fellow students engaged in a similar task. Discussion helps to clarify his ideas and also his plan of procedure. When he comes to the end, the finished achievement takes on all the splendor of success. It embodies all he has felt and lived during the time it has taken to complete. This is real experience. It is culture acquired through individual development and through collective cooperation. It is no longer school - it is life (Parkhurst, p. 219, 1922).

Under this plan, although the students were able to gain a perspective of the whole year's work, they usually 'contracted' only a month's work at a time. There were no school timetables so that each student in cooperation with a teacher-advisor constructed his own. His timetable was made so that he spent proportionately more time on the subjects in which he was weaker. Teachers were constantly checking to see that students did not forge ahead in subjects they liked at the expense of subjects they disliked; yet each pupil made his own choice as to the order in which he would study his courses daily. The rooms in the school were assigned to subject specialties, and a teacher interested in that subject was on hand to help students whenever required. Students were free to enter and leave the rooms at will, and often

gathered in small groups in certain rooms to study that subject. Graphs were kept by students and teachers to record the students' progress, and marks were assigned on the basis of percentage of work completed.

The plan depended upon converting a whole school to the contract method, and expected that there would be a specialist available in every subject area. It made no allowances for variable curriculum but merely expected weaker students to work longer at mastering the basic knowledge required for a particular subject.

C. C. Washburne, armed with the theoretical ideas of Dr. F. Burk and with the impetus of the citizens of the city of Winnetka, initiated the Winnetka Plan. He felt that the lock-step system was the cause of dissatisfaction with the schools; that only a very few students could proceed naturally at the pace which was arbitrarily set; and that the schools were constraining the better and frustrating the weaker students.

Under his guidance, a year's work was defined as: "what the slowest normal diligent child could accomplish in one year". Instruction was mainly through self-study materials (which the teachers wrote) and ingeneous games and group activities. Initially textbooks were also used, but these were phased out as the teacher-written materials became more complete. Emphasis was on making learning an exciting experience for the students. Teachers tried to make all know-

ledge learned as relevant as possible by taking an interesting situation and then examining it in great detail - extracting every piece of relevant information and taking the opportunity to introduce lessons expanding this information.

When a student felt he had learned a unit of material, he could do a set of 'A' exercises. These were complete with answers, and if he got any questions wrong he was to try the 'B' exercises. Then, similarily, the 'C' exercises. Completing this, he would then try a practice test, going through forms 'A', 'B', and 'C' if he had any errors. He was now ready for the 'real' test. Tests were not accepted if they had any errors, so that he might have to write another form of the real test more than once. Every student had to achieve 100% before he could go on to new work, creating a proper attitude of diligence in the students.

Washburne collected studies on the readiness of students, and took great care in establishing the curriculum for the school. Considerable use was made of intelligence tests in determining what a particular student should be studying, and it was the responsibility of the teachers to chart each child's course. Teachers also accepted the responsibility of creating new material, updating the old material, and coming up with ingeneous games and group activities.

As with the Dalton Plan, an underlying assumption here was that every student could learn every concept: it just takes longer for some than for others. Washburne makes no

mention of the difficulties that must have been encountered with the students working at widely separated levels in the same subject, and he found no faults with his teachers as they accomplished their many tasks.

2. TEACHING MACHINES AND PROGRAMMED INSTRUCTION

Teaching machines are devices which tutor a student without assistance from a human instructor. Three individualization criteria are met by all teaching machines:

1. They require the student's active participation.
2. They allow variability in the rate of each student's progress.
3. They provide the student with immediate knowledge of results.

The variety in style and purpose of teaching machines is only limited by man's ingenuity. The success of a device as a teaching machine depends upon how well it meets the above three criteria, how well it meets additional criteria desired for the particular machine, and pragmatic factors such as cost, ease of use, adaptability, etc.. The additional criteria proposed for a given machine usually depend upon the designer's theory of learning. A major dichotomy exists between the followers of Pressey and the followers of Skinner. Very simply, Pressey favoured the use of multiple choice devices where Skinner believed that users should construct their

own responses.

Pressey argued that test writing is a learning situation; its main drawback being the delay between answering a question and knowing the correctness of that answer. He also felt that a student should be able to respond to a question until he got it right, thus causing the greatest possible reinforcement between the question and the correct answer. Shortly after 1915, Pressey devised his punchboard which met the above criteria, those of a teaching machine, and was also cheap, adaptable, easy to use and easy to manage.

> This is a very simple unit the size of a three by five card, about one-half inch thick. A center of quarter-inch ply-board has riveted, on either side, two thin sheets of pressboard. Between the outside sheet and the second sheet is sufficient space for a slip of paper to be inserted; between the second sheet of the pressboard and ply-board center is a removable key sheet of pressboard. On each face of the punchboard are two columns each of four rows of one-eighth inch holes, the rows of four holes being numbered from one to thirty. However the key sheet has holes only for the right answer on the particular test to be used. ... The testee takes a test by simply punching with a pencil point through the paper slip in that hole which corresponds to what he thinks is the right answer. ... If he is right, his pencil goes through the paper and down into the hole in the key sheet. But if he is wrong the pencil barely breaks the paper and then comes up against the key sheet. He thus knows that he is wrong and tries another hole in (the same row), thus proceeding until his pencil goes deep (Pressey,pp.70-71,1950).

Although the above description is in terms of a test scoring device, Pressey did use it as a teaching machine. The program for the machine was a set of multiple choice questions. Experimentation over a wide range of subject matter including nonsense sylables, foreign language, technical

terms and psychology led to conclusions that units of thirty or fourty questions, discussed after try-out with the punchboard were better than longer units, and that four choices for each question were better than either true-false questions or questions with five or more choices.

A student may know he has answered a question wrongly, and even what the right answer should be without understanding the concepts involved. Pressey's answer to this objection was the above mentioned discussion period; he felt it shold be included with every lesson.

The work of Crowder however, led to a different answer - the scrambled textbook. Crowder reasoned that:

> It is worthwhile to point out that the process ... is different from that of simply furnishing 'knowledge of results' to the student. The test result is used to control the behavior of the teaching machine and need not necessarily be furnished to the student at all. ... The primary purpose is to determine whether the communication was successful, in order that corrective steps may be taken by the machine if the communication has failed (Crowder, p. 288, 1960).

Crowder went on to point out that the question of whether the child got the question right or wrong might not supply the most relevant information. Decisions can be made upon which 'right' answer a student chose, or which 'wrong' one.

> The simplest device which I will call a 'TutorText', is a specially prepared book in which each answer choice is identified with a page number. The reader choosing a particular answer, turns to a page number given for that answer. There he will find either the next unit of information and the next question, or, if the answer he chose was incorrect, he will find the correctional material appropriate to the answer he

> chose. He will then be referred to the original choice page to try again. The page numbers in the book are assigned essentially at random, and the reader therefore, cannot make progress from one page to the next except by making an active choice of an answer (Crowder, pp. 286-287, 1960).

Crowder also developed a device which he called the 'Tutor'. It worked in the same way as the TutorText, except that the 'pages' were on 16mm film. The student responded to a question by pushing an appropriate button and the next page was positioned automatically. Provision was also made for running the film to provide motion pictures.

The concept of branching, or directing the students' activity through a function of his responses is called intrinsic programming. Its development by Crowder was one of the major breakthroughs in the concept of teaching machines, and a definite advancement of education through technology.

Skinner, through experiments in conditioning animals, developed a theoretical framework which suggests that students should construct their responses rather than merely choosing one from a list of answers to a question. He took this as one of the main criteria for his teaching machines.

One of his earlier machines is often called the 'write-in' machine. The student is given a sentence with missing words, and he writes what he feels the correct replacements should be on a slip of paper exposed by the machine. Turning a crank at the side causes his writing to slide under a glass and the correct answer to appear. He then grades his response himself. After completing a cycle

through the machine, all questions which he graded as incorrect are asked again. An objection to this machine, that the student must be given the answer to check his response, is overcome in Skinner's second design. This is sometimes called the 'slider' machine.

> The device consists of a small box. ... On the top surface is a window through which a question or problem printed on a paper tape may be seen. The child answers the question by moving one or more sliders upon which the digits 0 through 9 are printed. The answer appears in square holes punched in the paper tape upon which the question is printed. When the answer has been set, the child turns a knob. If the answer is right, the knob turns freely and can be made to ring a bell or provide some other conditioned reinforcement. If the answer is wrong, the knob will not turn. A counter may be added to tally wrong answers. The knob must then be reversed slightly and a second attempt at a right answer made. When the answer is right, a further turn engages a clutch which moves the next problem into place in the window (Skinner, p. 110, 1954).

This machine has also been constructed with alphabetic characters instead of numbers on the slides.

A distinct advantage of the slider machine is that it requires more than just active participation on the part of the student; it demands a personal commitment. This commitment fosters both attention and motivation as factors of the learning situation.

The factors of intrinsic branching and construction of responses are difficult to assimilate into one machine. Because of the nature of our language, for any question there are usually several correct responses and an almost unlimited number of incorrect ones which can be constructed. The the ability to recognize all of these answers is beyond the

scope of any simple machine and therefore if these concepts are to be resolved it will have to be done by some sort of computer.

3. COMPUTER ASSISTED INSTRUCTION AND INSTRUCTION MANAGEMENT

The first publication describing the use of a computer as an instructional device originated at IBM (International Business Machines) in 1959, but before 1962 at least four major companies had set up experimental projects. (These were IBM; Bolt, Beranek and Newman Inc.; Coordinated Science Laboratories; and Systems Development Corporation.) All are still active in this research today.

The technique of time-sharing has been the greatest single contributor to the concept of computer assisted instruction. Using time-sharing, every student can interact with the computer as if he were the only one to whom it was 'listening'. In this context, the typewriter was the first device used in communication with the computer. Typewriters have been constructed which are relatively easy to connect to a computer; they are easy for students to use, but are often noisy and slow when required to print out large amounts of material. (A normal person can read much faster than a computer can write on a typewriter.) Use of an oscilloscope led to a more generalized cathode-ray tube system which could display information much more quickly and quietly. New de-

velopments include closed circuit television to project slides and moving pictures, audio systems to play and record sound, and complex accounting systems which can record a students progress every hundredth of a second. This latter development is providing opportunities for pedagogical and psycological research that were previously impossible. Further improvements in time-sharing have already made it less expensive to provide a computer, rather than a teacher, for a large number of students.

One of the major drawbacks of using the computer as a teaching machine is the cost of developing programs. It is relatively easy to enable the computer to provide drill for the students, but this can be done just as effectively by the teaching machines mentioned; and at far less expense. More difficult to construct are programs which tutor the student; but here, if anywhere, lies the future of computer assisted instruction.

Many intrinsic programs have already been developed which in a sense tutor the student, but nearly all avoid the construction of response technique or limit the acceptable responses severly. The technology for evaluation of random answers is far from complete. More pressing problems however, are the nature of computer languages and the geometric expansion created by intrinsic programming. Computer languages now available for programming are difficult to learn and tedious to write. It requires about one thousand

computer instructions to present a half-hour lesson to a student if no branching is allowed. When branching is included, these programs expand without bound as every eventuality of student deficiency must be pre-empted and an appropriate segment written to instruct the student. The initial organization of these branching programs is in itself an enormous task; generalities such as 'educating for democracy' and 'educating the whole child' have no meaning to the computer. It must be told what, when and how to teach each student each new idea.

On the credit side, it can be noted that new and more versatile languages are being developed. (For example; VAULT developed by the University of Alberta in cooperation with IBM.) Further, once a computer program has been written it can be used by an unlimited number of students. It is also amenable to adaptation and extension. But the most valuable asset of the computer-teacher is its ability to record, in minute detail, the actions of each and every student. This information can be produced in such quantities that the computer must again be used to analyze it. Results of these analyses are being used to evaluate the computer programs, the students, and pedagogical and psychological theories of learning in the detail that is necessary to their thorough understanding.

While the advent of wide-spread computer assisted instruction is still in the future, computers are being use-

fully employed as management assistants to the teachers. In this role, the computer is used not to instruct the students, but to help the teacher by relieving him of many of his administrational duties and enabling him to concentrate more of his efforts on the actual teaching process.

The computer is capable of storing large quantities of information, and then supplying it whenever required. Banks of test questions; lesson plans; weekly, monthly, yearly plans; specific laboratory plans; information on reference books; projects that could be undertaken and modes of teaching that could be employed are among the data that could be stored. By typing in key words at a computer terminal, the teacher could access any of this material whenever it was needed. Statistics on test scores, class averages, test analysis, attendance, etc. are easily calculated by the computer and stored for quick reference.

With many of the new projects being tried in individualizing instruction, a great deal of evaluation is indicated. This job normally falls to the teacher, with the result that he becomes almost peripheral to the education of the student; and many of the good effects that might have been realized are lost. Perhaps with the computer to assist in these time consuming chores, more progress will be made in designing practical means of individualizing instruction.

4. SOME RECENT INDIVIDUALIZATION OF INSTRUCTION PROJECTS

Although many of the recent individualization of instruction projects meet the three basic criteria of teaching machines, they do constitute a noticable departure from the teaching machine concept. While relying on some form of programmed individualized instruction, they try to meet other needs as well.

The conceptual difference between these two modes of teaching is contained in the distinction between training and education. In the words of Glaser;

> ... the distinction between training and education is usually made in two ways. a) The specificity of the behavioral end-products. When the end-products of learning can be specified in terms of particular instances of student performance then instructional procedures can be designed to directly train or build in these behaviors. When the end-product behaviors cannot be specified precisely because they are too complex or because the behaviors that result in successful accomplishment in many instances are not known, then the individual is expected to transfer his learning to the performance of the behavior which was found difficult to analyze. Obviously, schools do not train for every instance of behavior that will occur in the future, but they rightly expect that individuals will generalize or transfer their behavior to similar and novel instances. A distinction between training and education being made here is the amount of transfer involved and the precision with which the behavioral end-products are specified. If the end-products of the learning process can be rather precisely specified, as, for example, learning to use the slide rule, then it can be said that the student is being trained to use the slide rule. On the other hand, if the behavioral end-products are complex and present knowledge of the behavior makes them difficult to specify, then the individual is educated by providing a foundation of beha-

vior which represents approximations to the behavior it is wished that the student will eventually perform, e.g., being a creative scientist. b) Minimizing vs. maximizing individual differences. A second distinction between training and education had been referred to above. This is related to the fact that training with reference to specific behaviors implies a certain uniformity. Individuals are taught to perforn similar behaviors, and they learn to do so within the limits of the variability introduced by individual differences. Education, on the other hand, attempts to maximize individual differences by teaching in such a manner that each individual eventually behaves in a way singular to him on the basis of the groundwork of a basic education. As a result, individuals learn to create, invent, and solve problems in non-uniform ways (Glaser, 1962).

Teaching machines, and to a large extent computer assisted instruction, are directed almost entirely at the training level. This occurs by definition as the desired student behavior must be specified in advance. Teaching machines and the presently programmed computer-teachers cannot recognize creativity by their very nature.

The projects discussed in this section provide reasonable solutions to this problem by attempting to individualize instruction without ignoring the need to provide channels for personal communication between the student and other humans. Four recent experiments are presented: PLAN (A Program for Learning in Accordance with Needs), I.P.I. (Individually Prescribed Instruction), I.M.U. (Individualized mathematics instruction), and an experiment conducted by Mortlock.

About thirteen years ago, CREATE (Center for Research and Evaluation in Applications of Technology in

Education) initiated a program called project TALENT. Their work led to project PLAN. Project TALENT identified four basic needs in the current education system, viz;

1. A need to provide for the very large individual differences found in any grade or age group.

2. A need to assist the student in:

   a) developing a sense of responsibility for his educational, personal and social development, and

   b) making realistic educational decisions to make full use of his talents.

3. A need to broaden the focus of educational objectives in order to:

   a) facilitate occupational planning,

   b) emphasize responsibilities of citizenship, including personal and social development, and

   c) include aspects of a general education which will help students make more satisfying use of their anticipated increase in leisure time.

4. A need to provide more flexible and efficient curricula and instructional methods necessary to enable each student to plan his education to prepare him for the roles he selects.

Project PLAN is an attempt to overcome these deficiencies through the use of computer assisted instruction management (although they apply the term computer assisted instruction). The implementation of PLAN required the addi-

tion of five new components to the present educational system.

1. Detailed educational objectives. These objectives would be listed, as far as possible, in terms of behavior changes expected. They would not be considered exhaustive, nor would they remain fixed. Follow-up studies and detailed case studies of students would provide direction for periodic revisions.

2. Procedures for measurement. One of the gaps in the current system is an adequate means for measuring the important objectives of education.

3. Teaching-learning units. Several teaching-learning units varying in content, mode of presentation, length, and/or some other dimension would be prepared for each set of objectives, so that at least one of them could be expected to provide proper instruction for any student who has completed the prerequisite work.

4. Methods and materials for guidance. For the student to choose his own educational objectives, and for the teacher to be able to help him, methods and materials must be adapted or developed to evaluate the contingencies.

5. Technological equipment and computer programs. Essential to the functioning of the whole system is a computer which would assist the teacher by performing the scheduling, scoring, record keeping, and information storage and retrieval duties.

The computer would compile detailed information on how each student learned, the goals he had achieved, his interests, and his estimated potentials. At the beginning of a school year, a synopsis of a particular student's record would be given to a teacher, who, in conference with the student would set up long, intermediate, and short range goals. Input to the computer, these goals would cause teaching-learning units matched to the student's learning style and aptitudes to be produced, along with suggested readings, workbooks, audio-visual material references, and specific suggestions. Progress reports, input regularly, would enable the computer to construct periodic tests and update the student's records (Mager, 1967; Flanagan, 1967).

Four objections to project PLAN arise:

1. It may be morally objectionable to have such detailed information about people stored for ready access in a computer.

2. A teacher's classification of a pupil, entered into the computer might cause a stereotyped approach to his learning.

3. A system such as the one envisioned would be costly to undertake and maintain.

4. It may turn out that the teacher, with his limited knowledge about each student, is still able to make better educational decisions concerning materials for a child than a computer can.

I.P.I. is a program of individualized instruction developed by LRDC (Learning Research and Development Center of the University of Pittsburg) and disseminated by RBS (Research for Better Schools Inc., Philadelphia). It has been implemented in over one hundred schools in both Canada and the United States, mainly in the areas of elementary school reading and mathematics.

Under I.P.I., the teacher's role would be diagnostic and prescriptive rather than mainly instructive; he would spend much of his time reviewing student marks and making out prescriptive sheets. These sheets would be used by students to locate their current assignments. Behavioral objectives would be used, and the students tested often (the tests would be marked by teacher aides) to make diagnostics as accurate as possible. The objectives would be grouped in streams of content and could cross up to seven levels of depth. Lesson materials would be geared closely to the objectives and the teacher could often prescribe at least six settings of presentation (teacher tutor, peer tutor, small group, large group, seminar, or independent study). Up to five distinct sets of materials (texts, films, records or tapes, research, or manipulative devices) would be available for any given lesson.

Six important factors present in the I.P.I. system are:

1. Each student determines his own rate of learning.

2. Each student determines his required amount of practice.

3. The nature of each student affects his learning situation.

4. Students come to expect mastery (85% is required).

5. Students can work independently.

6. Children are guided daily by a teacher.

Some of these factors are not directly related to individualized instruction, but are important none the less. The usual forms of evaluation cause students to take 'beating the class average' as a learning goal. This makes evaluation superflous and incidentally causes children in competition to hoard information, rather than sharing it with classmates. Expecting mastery of behavioral objectives has the opposite effect, causing children to cooperate and making evaluation simpler for the teacher and more relevant for the child.

Having students work independently is worthwhile not only because it allows them to work at their own rate, but because it fosters a sense of responsibility for one's own learning and causes students to develop their own goals. Giving students an individually prepared prescriptive sheet daily is also a considerable advantage, especially at the elementary level. It gives children dignity and a sense of their own self worth.

I.M.U. was conducted by Oreburg under the Institute of Educational Psychology, Malmo School of Education in

Malmo, Sweden. The project began with pilot studies in 1963 and was slated to conclude with the main experiment on 11,500 pupils in 1971.

> The pilot study showed, among other things, that self-instructional material should be graded in terms of degree of difficulty and divided into small units. ... Moreover, it showed that the pupil's working capacity showed great variance. The fastest pupil worked through a certain material about 10 times more extensive than that which the slowest pupil could cover (Oreburg, pp. 3-4, 1968).

Figure 1 shows the ways in which a student worked through the material developed for I.M.U.. Component 'A' is common to all students, but every student works at his own pace. On completion of component A, each student took a test and then selected one of components B(1), B(2), B(3) with the help of his teacher. (The percentages in the diagram indicate the usual distribution.) Finishing this (each component took from three to four weeks), students wrote another test and then selected a 'C' component. A final test; and for those who needed extra work, the D component; finished the module.

The components with subscript '1' were intended for the weakest students, and subscript '3' for the best. These were similar in nature, but contained work varying in degree of difficulty. The dotted lines in the figure show that the students sometimes changed levels during a module; but at regular intervals they came together again in the A component.

The D component contained material for repetition

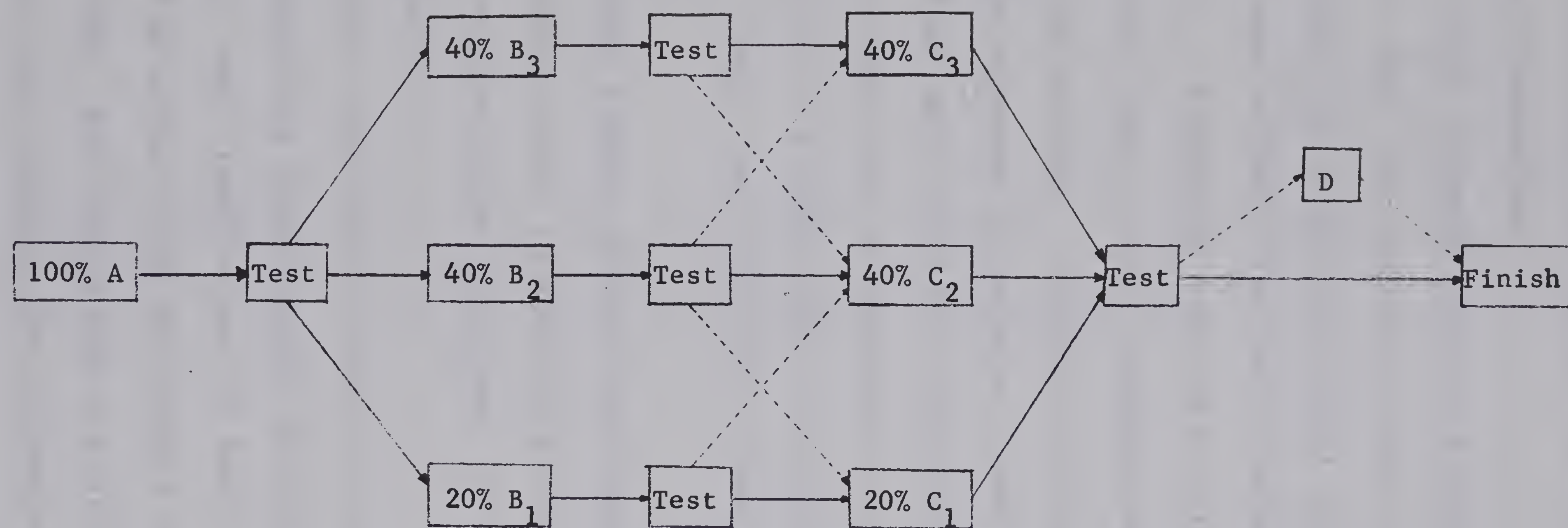

FIGURE 1

THE LOGICAL FLOW OF AN IMU MODULE

in connection with diagnostic tests, individual work, group work, and group discussions.

The I.M.U. program has two distinct differences from the I.P.I. One, the teachers using I.M.U. were obliged only about ten times per school year to consider which material was suitable for each student, as opposed to daily with I.P.I. Secondly, the I.M.U. allows for variability of content; that is, all students were not expected to cover the same content. Students constantly working at level three were exposed to material involving more depth of understanding than were the level one students. This consideration is perhaps not as important at the elementary level (where I.P.I. is used), but is certainly relevant in higher grades (I.M.U. was used with grades seven, eight and nine).

Mortlock (1969) devised and conducted an experiment in individualization in 1967-68 at University High School of the University of Michigan. His initial plan is depicted in figure 2.

Students were given a detailed set of terminal and subordinate objectives (constructed according to the guidelines of Mager), each labeled one of Basic, Intermediate, or Advanced. The whole class was taught the section at the intermediate level, and then given a test (Phase I). The results of this test streamed the students into three groups: Basic, Intermediate, and Advanced. These groups

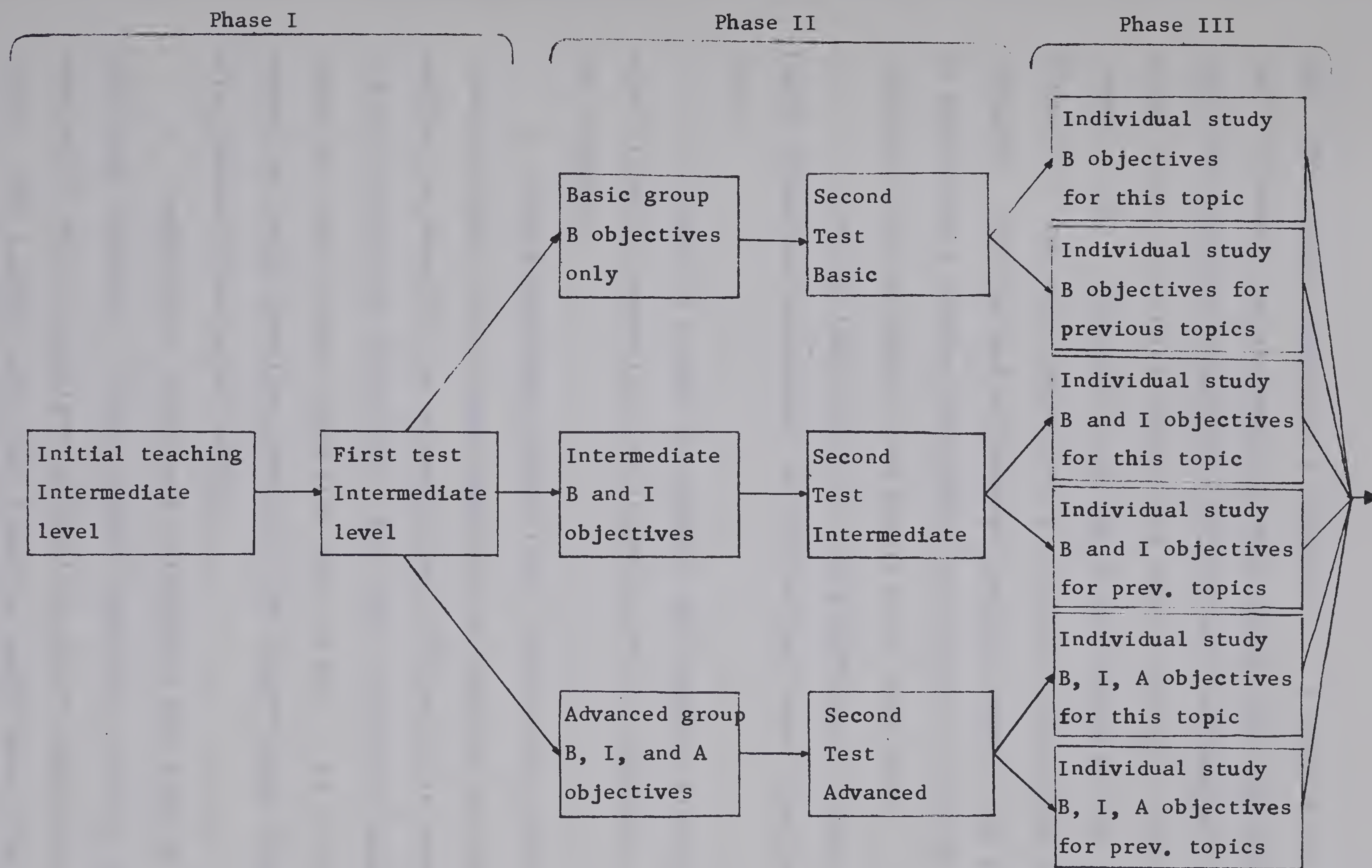

FIGURE 2

LOGICAL FLOW OF MORTLOCK'S UNIT (INITIAL DESIGN)

were then taught separately (Phase II). A second test further streamed the students of each group into two categories; both studied unachieved objectives independently and wrote tests on single objectives when they were ready (Phase III). On expiry of the time available for Phase III, the next topic was started and the three phases repeated. At the end of several topics, comprising a unit, an end-of-unit review assignment was given to provide the students with an overall perspective of that unit.

During the course of the experiment, three significant classroom management modifications were found necessary:

1. Phase III used up student and teacher time without significantly improving the learning of the students. It was dropped.

2. The subordinate objectives included had several disadvantages: they slowed teaching by causing too much attention to detail, the students had difficulty in recognizing the important objectives, and their inclusion made the tests too long (every objective was tested). As a consequence of these problems, many of the subordinate objectives were removed from later topics.

3. Independent study was introduced for some of the more able students. They reacted favorably to this, and were still able to achieve objectives at an acceptable level. The teacher was then freed to spend more time with

the less able students.

Two other classroom management contingencies seemed important as judged by the students on questionaires.

1. The listing of objectives was important to the students. Students felt that the list was motivating, because it made the course organization apparent, it translated the text to a practical level, and it served as a good reference for homework and review. This list also helped the teacher by making tests and lessons easier to prepare, and giving precise indications of student weaknesses. (Some of this latter advantage was lost when many of the subordinate objectives were removed.) Students did not feel that they fragmented the course, but did feel that since what was required was clearly laid out, it was easier to make up work missed through absenteeism.

A final point is that a teacher who was to assist Mortlock had difficulty in adapting to the situation. Mortlock judged this to be a result of the experimental environment being imposed from without; he suggested that this problem could have been avoided if the teacher was allowed to enter into the designing of the experiment.

There are many more projects concerned with individualized instruction that were recently or are currently active. The four chosen here were presented because each had some significant ideas that contributed to the construc-

tion of the current experiment. Project TALENT, in identifying the four basic needs of our present educational system, has provide an overview of the problem. Project PLAN's solutions have shown a general method of attack. I.P.I. has been presented because of its organizational concepts, and also because it is currently being tried in Edmonton schools. Through its solid funding and general availability it has quickly become an attractive vehicle for experimentation in both Canada and the U.S. The presentation of I.M.U. has shown that our problems are not regional, has acknowleged a similar management plan, and has contributed a number of significant ideas. The overview of Mortlock's experiment has brought out the final concepts necessary to the historical basis of the present experiment.

Any experiment usually has two guiding forces; one is the successes and failures of similar experiments in the past, and the other is the hypotheses and theories about the variables being considered. The former has been considered in the present chapter, the latter is the topic of chapter III.

CHAPTER THREE

THEORETICAL RATIONALE

It was the purpose of this experiment to devise and test a form of instruction for grade seven mathematics which would meet the individual needs of pupils in a more complete way than traditional instruction methods. In designing the treatment for this experiment, a number of basic assumptions and decisions were necessary. The purpose of this chapter is to present these basic assumptions and to rationalize the more important decisions that were made.

The first section of this chapter presents a list of student variables which led to the decision to incorporate four broad curriculum aspects into the treatment design. The following three sections discuss the rationale behind the organization of the treatment, the use of ability grouping, and the construction of behavioral objectives.

1. A BASIS FOR INDIVIDUALIZATION

The Hall-Dennis report makes over two hundred and fifty specific recommendations to various bodies concerned with education in Ontario. However,

> All of them are designed to support the one fundamental recommendation of this committee:
> Establish, as fundamental principles governing school

> education ...
> a) the right of every individual to have equal access to the learning experiences best suited to his needs, and
> b) the responsibility of every school authority to provide a child-centered learning continuum that invites learning by individual discovery and inquiry (Hall, Dennis, et. al., p. 179, 1968).

The first step in designing a program to meet the needs of students is to identify the variables which may influence the fulfilling of these needs. This experiment was concerned with a single need; the need of grade seven students to learn appropriate mathematics. Following is a list of variables thought to be important to this need by the experimenters.

1. INTELLIGENCE. The general intellectual ability of the child.

2. APTITUDE. Specifically, aptitude for mathematics. This includes computational skills, ability to reason logically, and ability to use symbols.

3. LEARNING STYLE. The ways in which a student is best able to learn.

4. CREATIVITY. Used in its broad sense, but most importantly, the ability to arrive at problem solutions which have not been specifically taught.

5. STUDY SKILLS AND HABITS. The effort applied and the attitude of students, to homework and self-directed study.

6. READING ABILITY. This includes reading speed, comprehension, and vocabulary.

7. PRIOR KNOWLEDGE. The knowledge that a student has

available to bring to bear on mathematics learning. This refers mainly to mathematical knowledge.

8. RESPONSIBILITY. The ability of a student to hold himself accountable for his own actions, especially his own learning.

9. MOTIVATION. The internal and external forces acting on a student which cause him to learn.

10. INTERESTS. Those fields of study or endeavor in which the student has knowledge or a desire to learn.

11. ATTITUDE. The student's feelings towards school, towards mathematics, towards peer groups, and towards competition.

12. ANXIETY. The student's apprehensions about school, about new situations, and about evaluation of his efforts.

13. ATTENTION SPAN. The length of time which a student can attend to one particular activity.

14. PHYSICAL CONDITION. Those aspects of a student's health, of either long or short term duration, which affect his ability to learn.

Although these variables overlap in many important areas, this does not detract from their usefulness as guidelines in designing the treatment. The interaction of these variables within students can be observed to result in three main effects.

1. Some students are able to learn more quickly than others.

2. Time being equal, some students are able to learn a given concept more thoroughly than others.

3. Ability being equal, different students may learn better through different modes of presentation.

The first two of these effects are closely related. The student who is able to learn quickly is also usually the one who is able to learn most thoroughly. The variables which seem to result in these two effects are coincident. Long term variables, those which remain relatively stable for each student over a month or more, which may influence these effects are: intelligence, aptitude, study skills and habits, reading ability, prior knowledge, motivation, attention span, and physical condition. The last three of these; motivation, attention span and physical condition; have minor fluctuations from day to day that can considerably influence the functioning of the long term variables. This effects the speed and thoroughness of a child's learning.

Provisions can be made for these two effects by allowing the amount of time each student has for learning a given concept to vary. Besides recognizing the variables underlying these effects, this provision may also result in a positive influence on the variables of responsibility, motivation, attitude, and anxiety.

Carroll (1962) feels that the time factor is most important in determining a student's degree of learning.

Thus:

Degree of learning = (time actually spent / time needed).

> The numerator of this fraction will be equal to the SMALLEST of the following three quantities: (1) opportunity - the time allowed for learning, (2) perseverance - the amount of time the learner is willing to engage actively in the learning, and (3) aptitude - the amount of time needed to learn increased by whatever amount necessary in view of poor quality of instruction and lack of ability to understand less than optimal instruction.
> This last quantity (time needed to learn after adjustment for quality of instruction and ability to understand instruction) is also the denominator of the fraction (p. 730, 1962).

The factors of perseverance and aptitude are allowed for by providing varying amounts of time for learning tasks; administrational allowances should be made to insure that opportunity is never less than either of these.

This creates a problem in that some students may have the perseverance and aptitude requiring an unacceptably large amount of time for a given learning task. The only solution appears to be to vary the learning tasks among students so that:

1. students will receive learning tasks which will not require an inordinate amount of time for them, and

2. students who consistently require large amounts of time will get fewer learning tasks.

This solution is a pragmatic one which recognizes the simple accepted fact that given the present school system, some students will learn more than others.

The variable of learning style is responsible for the third main effect: ability being equal, different students may learn better through different modes of presentation.

Many students can learn well through printed material, but audio-visual material and manipulative apparatus are clearly advantageous for some students. In recognition of this effect, opportunities should be provided for students to choose the mode of learning which best suits their own learning style.

Only two of the variables listed as influencing the learning of mathematics remain to be accounted for. They are interests and creativity. Because of their fluid nature, only a very flexible situation can allow for them effectively. Some form of enrichment activity offered in a loosely structured time interval would allow the teachers to: (a) provide the opportunity for students to persue personal interests related to mathematics; and, (b) foster and develop creative abilities in individual students.

The method proposed for meeting the need to learn mathematics in view of the related variables then, is to provide a variable curriculum. This variable curriculum must contain the following four aspects:

1. varying time for each student on each learning task,
2. varying learning tasks for individual students,
3. varying modes of presentation or learning, and
4. enrichment opportunities.

These four aspects cannot guarantee a fully individualized instructional program through implementation; nor do they circumscribe the possibilities for individualization.

Rather, they present four segments of a program which has the potential to allow for all of the variables listed as being important to students in the learning of mathematics.

There is also an implied hierarchy in the above listing. It is very difficult to allow variations in the learning tasks set for children unless provision is also made for students to work for varying lengths of time on these separate tasks. Therefore, varying time must be established before varying content can be implemented. Similarly, if students are to be allowed different modes of instruction, then the possibilities of varying content and varying time must also be available. Varying modes must come third.

In constructing a curriculum then, priority should be given to allowing varying content and rate. The implementation of varying modes is a difficult task, but a set of procedures could be devised which would allow for the addition of varying modes with less difficulty.

The fourth aspect, enrichment opportunities, although basically used to account for interest and creativity, could be used as a 'governor' in the curriculum. Students who worked very rapidly could be directed to enrichment work so that the time differential between students would not become unmanageable. This use of enrichment would serve a useful purpose, but is not altogether defensible. The fast-working students are not the only ones for whom enrichment opportunities should be made available. This objection can

be offset somewhat by insuring that all students have at least some time for enrichment work.

2. ORGANIZATION OF THE TREATMENT

The amount of responsibility a student is able to accept for his own learning is related to his perception of the organizational structure of the learning tasks. If the student is unable to determine how his learning tasks are arranged; if he cannot perceive which task to attempt on completion of the current one, then the organization is opaque. He cannot then, take the responsibility for going on to the next learning task. Further, this encourages the belief that the responsibility for directing his work and therefore the responsibility for his learning resides in the teacher, not himself.

A translucent organization is illustrated by the I.P.I. system of instruction (discussed in Chapter II). Here the student is aware that the results of his work will determine which learning tasks he will have next. Providing an obvious link between a student's work and his learning tasks encourages the student to appreciate his responsibility for his learning. However, since the teacher must make the actual decisions and assign the learning tasks, the student is not himself fully accountable.

One of the assumptions underlying the present ex-

periment was that students must be aware of the responsibility for their personal learning. Therefore a more nearly transparent organization was designed. This does not mean that students would be able to do anything they wished, but rather that all possible choices would be presented to them and they could then choose freely among the alternatives. This also does not mean that students would not be accountable for their choices, but rather that the consequences of their actions would be clearly spelled out beforehand.

A major implication of this decision is that the organizational structure and procedures must be clearly understood by all students. This can be done through an orientation period during which the instructional model and all alternatives open to the students are presented by the teachers. This orientation would proceed more slowly for some students than for others, and teachers would have to insure that each student's grasp of the organization of the whole treatment was complete.

The transparent organization of the treatment would not be observable in any one phenomenon; but would permeate the whole program. Students would be able to make their own decisions within broad guidelines. They would be able to choose certain topics, and omit others which they felt were beyond their abilities or which they felt they had already learned. Even though such a program would have a very complicated structure, it would be repetitive. As students went

through the procedures again and again, they would be able to grasp the whole import of their choices.

3. ABILITY GROUPING

> The school responds to differences in learning rates in many ways. Sometimes the policy of the school is, in effect to ignore these differences; a certain amount of time is provided for everybody to learn and no more. (For example, at some military academies, study time is prescribed and scheduled uniformly for all cadets.) At the opposite extreme is the case where each student is allowed to proceed at exactly his own rate; private instruction in music or foreign languages and self-instruction by teaching machine or other means are approximations to this case. The middle position is occupied by learning situations in which there is some kind of 'ability grouping': Pupils are assigned to different groups, classes, or curricula on the basis of learning rates (Carroll, p. 727, 1962).

A parallel situation exists with learning content. However, in most schools a certain content is specified for everyone, and this is what they are expected to learn.

Through a merger of self-instruction and ability grouping, it is possible to devise a program in which each student can proceed at his own rate and cover individualized content. Perhaps the easiest way of providing for self-instruction is through self-study printed materials. The most reliable method of ability grouping is based on previous achievement, especially previous achievement on the same content. Group membership will help to determine what content individuals should cover; and programming the self-study materials could make this aspect even more individually

tailored. Students should, of course, be able to proceed at their own rates.

It might be expected that faster learning students would enter the higher ability group. Here they could be given more learning tasks than the other groups to help remove the time differential between them and slower students. Similarly, lower groups could be given fewer than average learning tasks; however, care would have to be taken to remove only those tasks which were not essential to mastering content which might arise later in the program.

A basic assumption of the experimenters was that the treatment should be designed so that students could attain and would be expected to achieve mastery of the subject content they were given.

> Most students (perhaps over 90 per cent) can master what we have to teach them, and it is the task of the instruction to find the means which will enable our students to master the subject under consideration. Our basic task is ... to search for the methods and materials which will enable the largest proportion of our students to attain such mastery (p. 1)
>
> There are many alternative strategies for mastery learning. Each strategy must find some way of dealing with individual differences in learners through some means of relating the instruction to the needs of the learners. We believe that each strategy must include some way of dealing with the five variables discussed in the foregoing (p. 7). (These five variables were:
> 1. aptitude for particular learning,
> 2. quality of instruction,
> 3. ability to understand instruction,
> 4. perseverance, and
> 5. time allowed for learning.) (Bloom, 1968)

All students cannot be expected to master the same content, therefore at least three hierarchical content pack-

ages should be available. By virtue of the achievement group of any student, the content package he would be expected to master would be determined. The content for individual students would naturally overlap to a large extent (they are taking the same course).

Because of fluctuations in the short term variables discussed in the first part of this chapter, it cannot be assumed that student group membership would remain static. Mortlock (see Chapter II) found that students changed groups fairly often and also that students consistently in the lower group felt they were 'missing something' if they were not allowed to at least attempt some higher group tasks.

Many of these considerations can be resolved by providing a program with two 'phases'. The first would present all students with intermediate group level tasks, perhaps indicating which were essential and must be learned by the lower group. (This group would then have the option of ommitting tasks which were not imperative to them.) A test of mastery on the intermediate level tasks could then be used to separate the three groups. The second phase of the program would then consist of more directed work for the lower group, review for the intermediate group, and work on advanced tasks for the higher group. (A plan similar to this was used by Mortlock, see Chapter II, and Mortlock, 1969)

4. THE BEHAVIORAL OBJECTIVES

Three ideas are discussed in this final section: the need for behavioral objectives, a practical method for constructing these for grade seven mathematics, and the use of objectives in a variable curriculum.

Mager (1962, p. 4) describes a situation in which a group of students were given a series of courses, each with three exams. The students consistently scored poorly on the first exam and then improved considerably on the last two. A group of consultants attributed this phenomenon to the fact that students were unclear of the objectives for each course. They used the first exam to find out what the objectives were, and were then able to perform better.

Tyler (1964, p. 279) identifies three student sources of information on the objectives of a course: the textbooks and workbooks, the teacher's actions in class, and advice from other students. He goes on to point out that:

> Unless the exercises in the workbooks and the textbook assignments clearly reflect the desired objectives, the student is likely to resort to memorization and mechanical completion of exercises rather than to carry on the activities which are really relevant to the required goals (p. 280, 1964).

These points are clearly important if students are to proceed using self-study materials; as one of the main sources of objective definition, the actions of the teacher, would be removed. In a variable curriculum situation, where different students would have different objectives, it is

even more important that the textbook, workbook, or self-study materials clearly define the appropriate objectives for the student to him.

In the words of Bloom,

> The specification of the objectives and content of instruction is one necessary precondition for informing both teachers and students about the expectations (Bloom, p. 8, 1968).

The method outlined by Mager (1962) for preparing instructional objectives is outlined by three headings: terminal behavior, restrictions, and criterion. An instructional objective is a statement of educational intent. For it to be effective, it must not only communicate what the learner will be able to do (terminal behavior) after the instructional sequence, but also the conditions (restrictions) under which he must do it, and the way acceptable achievement of that objective will be determined (criterion).

A problem arises in attempting to specify the restrictions for objectives in grade seven mathematics. Putting these restrictions in precise mathematical terms often gives rise to such unwieldy statements that students can not be expected to understand them. In other cases, the mathematics necessary to describe the restrictions has not yet been taught to the students.

Since it is necessary that students clearly understand their objectives, a more practical form for stating them is needed. In a situation described above, students were able to determine the objectives for their course from a

single exam paper. This technique can be applied to grade seven mathematics by presenting each objective as a simple statement of educational intent followed by a sample test question. The objective can be made even more useful by supplying a complete solution to the test question and indicating how the answer would be marked.

Objectives stated in this way still rely on Mager's principles, but in a more informal way. The terminal behavior is indicated by the solution to the question; the restrictions are implied in the question itself, and the criterion is indicated by the way in which the question would be marked.

If examiners are careful to construct all test items so that they are parallel to the samples given with each objective, this method has a second advantage; since students will know the type of question to be asked on the exam, a lowering of test-anxiety should result.

The use of objectives in the designed program is discussed in the next chapter, and examples of constructed objectives can be found in Appendix IIIc.

Although this chapter has not outlined all of the available theory behind the procedures employed in the present experiment, the main areas of departure from traditional instruction have been identified and rationalized. The following chapter presents a description of the experiment. In detailing the actual procedures used, much of the remaining theoretical background is implicitly clarified.

CHAPTER FOUR

DESCRIPTIONS OF THE EXPERIMENT

> Once a research study has been effected, the researcher describes it as carefully and completely as he can. ... other persons interested and knowledgeable in that same field may then replicate the research themselves. Replication is only possible if the study has been described in sufficient detail for another researcher in a distant laboratory to undertake the same experiment in a comparable way (Frymier,p.52,1969).

It is the purpose of this chapter to describe, in "sufficient detail", the experiment as it was conceived by the experimenters; not only for the purpose of replication, but to provide information on the background to and procedures for the devised method of individualizing instruction. The ways in which the devised procedures were implemented and modified as a result of the experiment are reported in the next chapter. The description of the program is divided into three segments which deal with: the levels, the instruction management plan, and the materials.

1. THE LEVELS

Fundamental to the operation of the instruction management plan was the concept of three levels: Basic, Intermediate, and Advanced. These are sometimes abbreviated B, I, and A, and are used in three ways:

1. to distinguish levels of expectation,

2. to distinguish types of objectives, and

3. to name three groups to which students were assigned.

Although it may be true that most students can learn every concept given sufficient time, the goal of the experiment was not to have most students learn every concept. Rather, it was to have every student learn a required minimum amount plus as much more as he was able to cover in the time available. For this reason, objectives were categorized as Advanced, Intermediate or Basic.

The Basic objectives were those considered by the teachers and experimenters to constitute the minimum level of achievement necessary to proceed sequentially through the course. In some cases, these objectives were indeed difficult to achieve; but difficulty was not the principle criterion of classification. The Intermediate objectives were those considered to constitute what the average group of students should learn; these were an overset of the Basic objectives. If an objective was felt to be desireable, could probably be mastered by most students, but was not really necessary to a sequential procedure through the course, it was classified as Intermediate. The Advanced objectives were those considered by the teachers and experimenters to constitute what the better students should attempt to accomplish above the intermediate level. Objectives found desireable, but which

average students could not be expected to achieve in a reasonable amount of time were classified as Advanced.

Many Intermediate objectives were constructed from Basic objectives by altering either the restrictions or the criterion for that objective. (An example would be objectives B3.1 and I3.1, see Appendix IIIc.) These objectives would be similar, and most often the B objective was restricted to numbers which caused relatively simple calculations. Objectives which were related in this way were numbered similarly (c.f., B3.1, I3.1). If a student achieved the I level objective in such a pair, it was assumed that he had also achieved the B level objective.

Students working at the Basic level were placed in the Basic group, and were expected to achieve all Basic objectives. Students working at the Intermediate level were placed in the Intermediate group, and expected to achieve all Intermediate level objectives; this included all Intermediate objectives and all Basic objectives for which there were no corresponding Intermediate objectives. Students working at the Advanced level were placed in the Advanced group and expected to achieve all intermediate level objectives plus most of the Advanced objectives.

Figure 3 may help to make these distinctions clear.

GROUP	EXPECTED TO ACHIEVE	WHICH CONSISTED OF
Basic	Basic Level objectives (what must be learned)	All Basic objectives
Intermediate	Intermediate Level objectives (what should be learned)	All Intermediate objectives and all Basic objectives for which there were no Intermediate objectives
Advanced	Advanced Level objectives (what may be learned)	All Intermediate Level objectives (as described above), and most Advanced objectives

FIGURE 3

THE THREE LEVELS

2. THE INSTRUCTION MANAGEMENT PLAN

No credit can be taken by the experimenters for invention of the elements of the plan of instruction management that was used in the experiment. The idea of dividing students into more homogeneous groups for purposes of instruction has been used by teachers for some time; independent study is a commonly studied, though perhaps not widely used educational technique; and the concepts of selective review based on a diagnostic test, variable curriculum, and self-pacing are also well worn educational ideas as the historical rationale has shown.

The management plan basic to the experiment uses all these ideas; perhaps not to their full extent, but as completely as the situation would allow. The following description is divided into four parts: the orientation, Phase 1, Phase 2, and other descriptions.

The Orientation

At the outset of the experiment, several class periods were to be devoted to explaining to the students what their role would be. Parents were also to be invited to spend an afternoon at the school to become familiar with the student's new routine (see Appendix II).

During the experiment, students were to take the responsibility for organizing their own activities, and therefore needed a very clear picture of the courses open to

them. They would periodically be given packets of self-study material. These packets were semi-programmed and highly structured. Instructions were given at every decision point in the packets, but student understanding of their general nature would be necessary to their effective use. Flowcharts (Appendix IIIa) would be provided for pupils to chart their course through the packets, and record pages (Appendix IIIb) were to be kept by each student to indicate the areas in which he needed work. The roles of these two devices would have to be clearly understood.

Students would be required to arrange for their own tests at the end of each phase. They would have to understand how to arrange for these tests, what the tests would cover, and when and where they could be taken. There were two tests for each topic, a diagnostic test called Test 1, and a final test called Test 2. The results of the diagnostic test were to be used to stream the students into the Basic, Intermediate, and Advanced groups. Pupils would have to know the criterion for streaming in advance, and also have to understand the levels concept discussed in the previous section.

The students would already be familiar with behavioral objectives from work done by the teachers prior to the experiment (see Appendix II), but they would now have to be familiar with the levels of the objectives being indicated by different styles of type and different numbering. Finally,

they would have to know the implications of being placed in one of the three groups, the limits on the speed at which they could work, and the advantages of working quickly and efficiently.

Naturally most students would not be able to grasp all of these ideas in a few class periods. Orientation was to be a continuing process and was expected to last well into Topic 2 of the experiment. For students who were unable to manage their own affairs after a reasonable amount of time, a special class was to be set up in which a teacher would tell the students what to do each day. This special class was planned so that as students became aware of what was expected of them, they could rejoin the main stream. Other students could join the teacher-taught class for one or two periods if it was covering something that gave them difficulty, or could be assigned to this class by a teacher who found it necessary because of discipline problems or an obvious inability to proceed.

Phase 1

Phase 1 of the management plan consisted of work on the Phase 1 Materials and Test 1. Samples of Test 1 are included in Appendix IIIe and samples of Phase 1 Materials make up Appendix IIIc.

The Phase 1 Materials for each topic were self-study packets of about one hundred and thirty typewritten pages. They were printed on selected colors of three hole

punched paper, and were divided into from ten to twelve sections. A more detailed description of their nature is presented in the next section of this chapter.

The function of the Phase 1 Materials was to serve as a common introduction to the subject matter of the unit for all students, and so it contained all of the Intermediate level behavioral objectives. The development of concepts, questions, and directions for activities were therefore aimed at the intermediate level. Students were expected to proceed through this material according to the directions they had been given during the orientation. The time used by each student was expected to vary, the fastest needing only five or six periods, the slowest using from ten to fifteen, and average requiring about eight periods to complete the Phase 1 Materials for a topic. Note that all students were to begin each topic working at the Intermediate level.

A form of pre-test was included in each section of the Phase 1 Materials. Some form of pre-testing was necessary for each student to determine what he should or should not cover, and it was felt that including the test items for each student's own evaluation would aid in developing his sense of responsibility for his own learning, and incidentally mean one less test for the teacher to mark.

When students had completed the Phase 1 Material, they were to arrange to take a post-test, Test 1. This test was constructed with one question for each intermediate level

objective contained in the Phase 1 Materials. Each question was parallel in difficulty and design to the examples given with the objectives, so that students would know exactly what they would be asked. Students would also know that achieving less than half of the objectives would place them in the Basic group, more than half but less than four-fifths would place them in the Intermediate group, and achieving four fifths or more of the objectives would allow them to enter the Advanced group.

Although these boundaries were to be known to the student in advance, they would not be inflexible. Students within one or two objectives of the boundary line would be able to appeal to the teacher to have their group designation raised. If the teacher agreed, this would be recorded on the student's record page, and he would then be regarded as a probationary member of the higher group.

Phase 2

Once a student had received his results and been placed in a group, he would be given his Phase 2 Materials. These were again self-study materials, but were different for the three different groups.

Students who entered the Basic group would receive materials which were concerned with the Basic objectives only. They would use their record page (Appendix IIIb) to tell them which parts of the Phase 2 Basic Materials to study. On Test 2, they would answer only questions keyed to Basic ob-

jectives which they had not achieved during Phase 1. (Recall that if a student showed achievement of an Intermediate objective which had a corresponding Basic objective, he was to be considered to have achieved that Basic objective as well.) The basic form of Test 2 contained only questions on Basic objectives.

Students who entered the Intermediate group would receive the Phase 2 Intermediate packet of materials which was concerned only with intermediate level objectives. Their record page would clearly indicate which Basic and which Intermediate objectives remained to be achieved. This packet contained less instruction than the Basic one, merely providing exercises on which the students could hone their skills. Students would be required to refer back to their Phase 1 Materials for the objectives and development of concepts, and then choose the particular exercises they would do. They would receive the intermediate form of Test 2, on which they would answer only those questions keyed to objectives which they had not achieved during Phase 1. This test contained questions for all Intermediate level objectives.

Students who entered the Advanced group were to be given the Phase 2 Advanced Material. It contained only advanced objectives. These were related to the objectives of Phase 1, but required more depth of understanding. Only the Advanced group were to be given the Advanced objectives. These students would still be responsible for achieving any

Intermediate level objectives they had missed, but these could number no more than two or three. The packets were assembled much like the Phase 1 packets, and were to be used for self-study. On completion they would receive the advanced form of Test 2 on which they would answer the questions relating to objectives they had missed in Phase 1, and all questions on Advanced objectives.

Phase 2 was to serve two main purposes. It would act as a review of ideas which students had missed during Phase 1, and it would allow variations in the amount of learning required of individual students.

Other Descriptions

Under the heading of other descriptions, four areas require further elaboration: topic-end activities, the record page, the tests, and the teacher-taught class.

(a) Topic-end Activities

Previous accounts (I.M.U., see Chapter II) of independent study have shown that great variation in the amount of time students take to complete a section could be expected. Under some circumstances, this would be desireable, but the experimenters felt that within the bounds of the present traditional school system, any attempt to reduce this phenomenon would be worthwhile. For this reason, a minimum pace would be set for work on each topic and teachers would monitor the work of slower students to see that they kept up. A maximum pace was also to be imposed by setting dates, before

which the next topic could not be started. Topic-end activities were to provide meaningful work for students who finished a topic before the next starting day. The two activities provided were called Non-routine Problem Solving and Enrichment.

When students finished Phase 2 of a topic, they would spend at least two class periods working non-routine problems. Problems were provided in the topic itself, but these were of a 'typed' nature in that they could be solved by a routine process that had been explained beforehand. The non-routine problems were chosen to require some creative analysis or synthesis.

Students were to receive these problems in the form of a booklet, the first part of which discussed the solving of problems in a general way and gave examples of problems and their solutions. The remainder contained about twenty problems divided into three sets and arranged in order of difficulty as judged by the experimenters. Answers were supplied to the questions, but the actual solutions or methods of working the problems would be available only to the teachers.

Pupils would initially be directed to attempt the problems in the order they were supplied, but later the better students would be encougaged to skip to set two or three and begin working the harder problems immediately. If a student arrived at an incorrect answer, he would be encouraged to try again. If he could not solve the problem after several at-

tempts, he was to be directed to the next problem. After trying all the problems in a section, he was to go back and re-try the ones he had missed. Students would not be expected to be successful on or even try all of the problems.

When a student had finished all of the problems he thought he could solve, he was to turn them in to the teacher who would correct them and record the results on the student's record page.

Students who finished their Non-routine Problem section before the starting day of the next topic were to be given the opportunity to go to the enrichment area. This was to be housed at the back of one of the larger classrooms, and was to be stocked with such activities as a 'hard problem' box, mathematical games and puzzles, several outlines for individual or group projects, suggestions for directed reading, and other designed activities. Students here also had the opportunity to become student helpers.

(b) The Record Page

For each topic, each student would have a record page consisting of a single sheet of paper on which the numbers of the objectives for the topic were listed and the criterion for group membership indicated. The sheet also had space for recording the results of work done on the Non-Routine problems, group designation, and teacher comments. Its main use would be to record which objectives had been achieved, which remained to be achieved, and the student's

mark summaries. The teachers were to keep duplicate copies of each student's record page.

For a given topic, each record page had short horizontal marks: in the B column opposite every Basic objective number for which there was no corresponding Intermediate objective, in the I column opposite every Intermediate objective number, and in the A column opposite every Advanced objective number. (See Appendix IIIb for an example of the record page.)

The record page was not to be used until students entered phase 2; but as soon as Test 1 was marked, the teacher aide was to fill in the teachers' and student's record pages indicating all achieved objectives by placing a check mark on the appropriate horizontal lines. The total number of objectives achieved indicated in which group the student was to be placed. The check marks would make clear which objectives remained to be achieved.

If the student was to enter the Basic group, he would look in the B column to see which objectives to work on. The teacher aide was to have placed a new horizontal line opposite each Basic objective number which had a corresponding intermediate objective, and checked it only if the intermediate objective had been achieved. A horizontal line in the B column which was not checked then indicated an area in which the student was to work.

The horizontal lines would already be placed pro-

perly for the Intermediate group students, so every unchecked line in either the B or I columns would indicate their areas of deficiency. Indications for the Advanced group would be similar except that these students would also have to attempt the Advanced objectives.

Space was available on the record page for each student to indicate which of the Non-Routine Problems he had attempted, and for the teacher to indicate which of these had been satisfactorily completed.

When students were ready to write Test 2, they would take their record page with them to indicate which questions they had to answer. The teacher aide would check the duplicate record page when marking the tests to be sure the students had done this correctly.

(c) The Tests

As soon as a student had finished a phase of a topic, he would notify his teacher to arrange for a test on the next scheduled day. The teacher would then contact the teacher aide to have the student's name put on the attendance list. The teacher aide was expected to supervise the tests.

For security reasons, all Phase 1 tests were constructed in three forms (form A, form B, and form C), and all Phase 2 tests in two forms (form A and form B). This entailed constructing nine separate tests for each topic. They were put on different colored paper so that they could be easily distinguished. The tests were as similar as

possible, having the same questions but using different values. Since students could not know which test form they would be given, they could not memorize the answers of a friend. They would already be familiar with what the questions would be like because they corresponded closely to the examples given with each objective.

(d) The Teacher-taught Class

The teacher-taught class was to be set up after the first week of the experiment, and run continuously for the duration. Students were expected to attend for one or more of the following reasons:

1. they had trouble understanding the structure of the experiment, i.e., they were unable to decide what to do next,

2. they liked being taught by a teacher better than doing independent study,

3. they were placed in the class by a teacher because they were causing discipline problems or because the teacher thought they were having difficulty (1) above, or

4. they were having difficulty with a particular concept and joined the teacher-taught class to listen to a teacher explain that section.

Unless a student was assigned to the class by a teacher, he could commence or discontinue attendance freely, with the exception that this must be done either before or after a class period. The subject for discussion of each class was to be posted according to objective numbers, so

that all students could quickly determine what was being taught. This posting would also indicate the 'dead-line' for completing that section of material for the independent studiers.

Teachers were to organize the teacher-taught class in whatever way seemed most appropriate to them. Since this class would have to take the same test as the independent studiers, care had to be taken to see that the same concepts were covered in the class as with the materials. This task would be made easier by the presence of the behavioral ob-objectives.

On the whole, a very free atmosphere was expected to prevail while students were doing their independent study. Four or five classrooms would be available during the periods for grade seven mathematics, and students were to be able to leave their regular classrooms to consult books, write tests, attend the teacher-taught class, do enrichment, or other individual work. Within the rooms, students would be allowed to form small groups of two or three to discuss common problems, and most were expected to do their independent study in this way.

3. THE MATERIALS

The term 'materials' is used to mean the typewritten packets that students used for self-study during Phase 1

and Phase 2. The purpose of this section is to describe the materials themselves, how they were used, and their relationship to the whole experiment. They are discussed in the order that a student would encounter them for a given topic.

On the starting day for a topic, students were to be assembled in their regular classrooms and the Phase 1 materials distributed. A short orientation talk would be given by the teacher to outline the students' duties. Students would place their materials in special binders, with the record page and flowchart in front.

The Flowchart

Students were to use their flowchart to guide them through the materials. It would serve as a table of contents as well as a place to record their activities during Phase 1. (See Appendix IIIa for examples of the flowchart.) Students would be instructed to use a heavy pencil or crayon to trace the paths they followed and to color in the boxes representing the sections of the material that they worked through. The path of students having little difficulty would stay close to the left side of the page, but when a student was directed to repeated exercises in order to improve his skill, his path would extend to the right side of the page. Teachers could then tell from a glance at a student's flowchart whether he was having difficulty. Closer inspection would reveal the particular nature of the problem and the specific objectives involved.

The Introduction

Pages were color-coded to further help students with their organization. At the beginning of each topic was a single pink page on which the title of the topic was placed. Following this was a single white page containing a few short introductory paragraphs.

This introduction was intended chiefly for motivational purposes. It discussed some of the main ideas that would be presented in the topic, and attempted to show how these ideas would be useful to the student outside of school. (See Appendix IIIc for an example of an introduction.) Whenever possible, the new words used in a topic were used in the introduction and underlined. An attempt was made to show how previous work fitted in with the coming topic.

The Sections

The work for each topic was divided into about twelve sections. Although each section was an independent unit of work, they would have to be covered in order. These sections followed the introduction, and each was composed of a list of objectives on pink paper, a development on white paper, and a set of activities and exercises on buff paper (see Appendix IIIc for some sample sections).

Each section then, started with a pink page. The list of objectives started on the back of the page so that in cases where only one pink page was needed the student could open his book to have the objectives for a section on

the left and the corresponding development on the right. The objectives were numbered according to a simple coding system. The first character was either a B or an I representing either a Basic or an Intermediate objective. (Advanced objective numbers began with an A, but were only encountered in Phase 2.) Next was an integer representing the section to which that objective belonged. A decimal point and a final digit indicating the number of the objective in the section completed the objective code. For example; B3.2 represented the second Basic objective in section 3, I4.1 represented the first Intermediate objective of section 4. Objective numbers which were the same except for their initial character (e.g.; B3.2, I3.2) indicated corresponding objectives; these were aimed at the same skill except that the I objective was expected to be achieved under more difficult conditions. (This point is discussed more fully in the first section of this chapter.)

The purpose of the objectives was to indicate as clearly as possible what the student should be able to do when he finished the section. Each objective was composed of three or four parts: a statement of the educational intent, an example of a question that could be asked to test achievement of that intent, a complete solution to the question, and when necessary, the criterion which would be used when the question was marked. Often, where stating the ob-

jective in precise mathematical terms would have made it difficult to understand for the students, exactness was overlooked and the example was expected to convey the proper intent of the objective.

The objectives were listed in the order in which they were discussed in the development, and students were to be instructed to look them over carefully so that they would know what skills they would need to master. All students were expected to proceed through Phase 1 at the intermediate level, but those who might not expect to achieve this goal were given an option in the development. The explanations which were intended for only the Intermediate or Advanced students were printed in italic type so that Basic bound students could skip these paragraphs.

The development in each section was written in an inductive way whenever possible. That is, when ideas to be presented were of such a nature that the student could be expected to know or guess them beforehand, opportunities for showing this were provided by asking questions which pre-empted the following explanation. Other questions would give students the opportunity for immediate practice of an idea just presented. Students would be asked not to write on the materials, but would be encouraged to commit themselves by writing the answers to these questions on scratch paper. The correct answers were placed at the bottom of the page. These questions were enclosed in boxes to sep-

arate them from the text of the development.

In general, diagrams were used whenever possible to illustrate concepts, and examples included with step-by-step explanations of the procedures and reasons for the operations. At the end of the development, students were directed to re-read the objectives for the section, do further study if necessary, and then go on to the first check-exercise.

The activities and exercises, on buff paper, formed the last part of each section. It was to be composed of a check-exercise, a number of activities and/or exercises, and a second check-exercise. The check-exercises contained a question on every objective at the Intermediate level. These questions were parallel in difficulty and design to the examples given with the objectives. Students were to use them to evaluate themselves. They could answer these questions, check their answers with the ones provided, and then make their own decision as to whether they had it right or not.

If they decided they had mastered all of the objectives at that point, they were to go on to the next section, indicating this on their flowchart. If, in their own judgement, they were unsure of any of the objectives, they were to continue on with the activities and exercises.

The activities and exercises were keyed to particular objectives so that if a student decided he needed

practice only in a certain area, he could quickly find the appropriate items. The activities and exercises were arranged in order, from those designed to recall the necessary basic concepts to those which approached the difficulty of the question which the student had to answer to show mastery of the objective.

Students who proceeded in this way were to answer a second check-exercise after they had completed the appropriate activities and/or exercises. This second check-exercise was essentially the same as the first, and again the student would have to decide for himself whether to go on to the next section. If his decision was negative, he would have the further options of looking at a reference book (indicated in the materials), or seeking help from a student helper or the teacher. Space was available for recording these eventualities on the flowchart.

The Topic Review

The final section in each topic was a topic review. One or two white pages contained a list of the key ideas from each section of the topic, and a vocabulary list. (All of the key ideas pages are included in Appendix IIIh.) Students would be made aware of this review section at the beginning of the topic so that they could use it for reference. The vocabulary list contained a short definition for each term and the number of the page where it was first introduced. The topic review also contained, on

buff pages, a set of review exercises. These exercises were designed to tie all of the ideas of the topic together, and students could use them to prepare for Test 1.

The Phase 1 materials were completed by the solutions. In some cases, these green pages contained only the number or word answers to the problems of the check, activity, or review exercises. However, the more difficult questions had complete solutions provided.

Phase 2 Materials

Students would enter Phase 2 when they had completed their Phase 1 materials, written Test 1, and had their results recorded on their record pages. Each would now be assigned to one of the Phase 2 groups and receive a packet of appropriate materials.

The Phase 2 Basic materials were the only ones to include a new flowchart. These students' record pages would tell them which objectives they were to achieve during Phase 2, and they could use their new flowcharts to quickly arrive at the proper page in the Basic materials to begin work. (An example of a Phase 2 Basic flowchart in included in Appendix IIIa.) The Basic materials were designed much like the activities and exercises part of the Phase 1 sections, except that they contained a separate section for each Basic objective. All pages were on yellow paper except for the solutions, on green, collected at the end. The packet began with directions to the student to

re-read the appropriate objective and description from the Phase 1 materials and then do the exercises corresponding to each objective he was required to achieve. These exercises were again ordered from those directed at recall through those intended to develop the level of mastery expected. These exercises were interspersed with comments directing the student's attention to specific points which might be causing trouble. 'Rules to Remember' were placed in capital print. Another check-exercise was included at the end of these exercises. If the student got any part of it wrong, or was still unsure of any of the skill he was expected to master, he was directed to the teacher for help. Following this, he would be provided with another set of developmental exercises. When finished his Phase 2 materials, the Basic level student would write the Basic form of Test 2 and then go on to try some of the Non-routine Problems.

The Phase 2 Intermediate materials consisted of a number of yellow pages containing exercises specific to each objective, and collected solutions on green paper at the end. Students were to be instructed to consult their record pages to determine which objectives they had yet to master. They would then use their Phase 1 flowchart to determine which of the pertinent sections of the Phase 1 material they had not done. After re-reading the appropriate objectives and development, they would complete these unfinished portions of Phase 1 and then go on to the appropriate ex-

ercises in their Phase 2 material. Having done this for all unachieved objectives, they would be ready to write the intermediate form of Test 2.

The Phase 2 Advanced materials were constructed much the same as the whole of the Phase 1 materials. They had the same color coding; and contained a list of objectives, a development for these objectives, and a set of activities and exercises with solutions. Students entering the Advanced group would be responsible for mastering all unachieved objectives from Phase 1, but they would be asked to do this on their own by consulting their Phase 1 material. The Advanced material did not contain any check-exercises. Students, on completing the Advanced material, would write the Advanced form of Test 2.

Appendix III contains samples of all the materials actually used in the experiment, and a quick overview of the ideas covered in each topic can be obtained from Appendix IIIh where all of the key ideas for each topic are presented. Examples of the Non-routine Problems can be found in Appendix IIIf, and Appendix IIIg outlines some of the enrichment materials.

The actual operation of the Hardisty experiment did not follow the above outlined plan exactly. The deviations that were observed are reported in the next chapter.

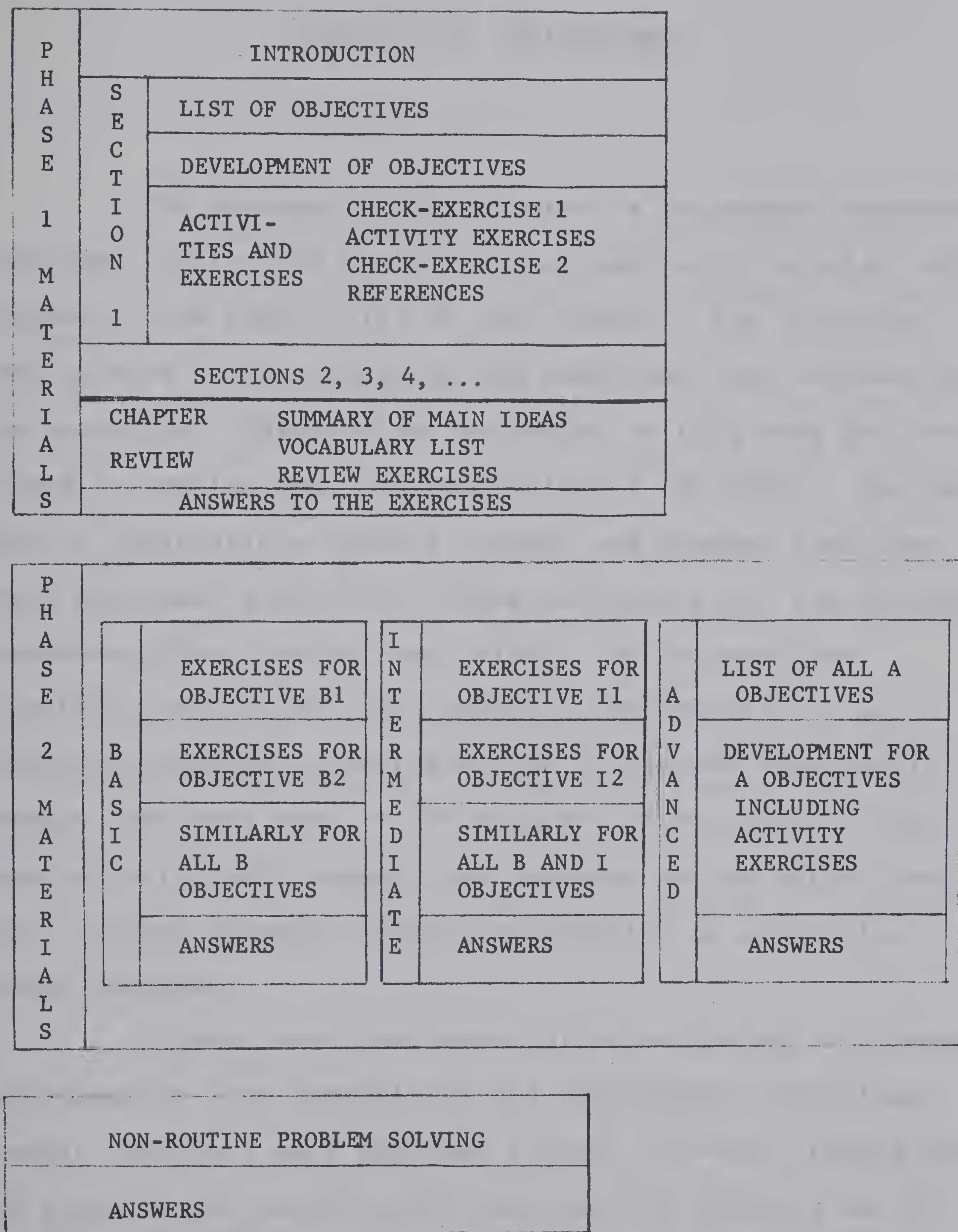

FIGURE 4

PHYSICAL ARRANGEMENT OF A TOPIC

CHAPTER FIVE

FEASABILITY INDICATIONS

The purpose of this chapter is to present observations made during the Hardisty experiment which provide indications of the feasibility of the program. The situation that existed at the outset of the experiment was relevant to its operation. Observations collected in this area are included in section one: Implementation of the Model. The main body of observations concern student and teacher reactions. These have been sorted into those pertaining to: the devised procedures (The Instructional Plan), the instructional materials provided for the students (The Materials), and classroom management considerations (Classroom Management). Changes that were made to the program during operation are reported with their causes, and comments by the writer have been included wherever further explanation or evaluation seemed necessary.

Student reactions were collected through a student questionnaire (see Appendix V) and spontaneous interviews. Teacher reactions were obtained through informal discussions and three formal meetings held between the teachers and experimenters (see Appendix IV).

The final section of this chapter summarizes the feasibility report. The observations presented in this

chapter lay the basis for the interpretations, evaluations, and implications presented in Chapter VI.

1. IMPLEMENTATION OF THE MODEL

As previously mentioned, students had been working with self-study materials written by the Hardisty Teachers prior to the treatment. This led to a positive 'set' towards the experiment by both the teachers and the students. Under conditions where this set was absent, care would have to be taken to insure a proper introduction of the program to the participants. It was felt that the teacher's appreciation of the effort involved in producing the materials for this study increased their commitment to the project. Because of the teachers' efforts, the students had developed an attitude of challenge or at least acceptance of the new teaching method. The Hardisty teachers had the foresight to invite the parents to several meetings to explain the new teaching method. This was continued under the experiment in an effort to stem criticism born of misunderstanding, and was largely successful.

A thorough orientation to the model by both teachers and students was considered very important. The model designed by the experimenters was substantially different from the one developed and used initially by the teachers, raising potential problems during change-over. During con-

struction of the model, the teachers were given progress reports. A number of preliminary meetings were held during which the teachers were invited to criticize the model. This procedure had at least four beneficial effects:

1. The teachers' critisizm helped to improve the model.
2. The teachers became familiar with the model.
3. The teachers became more committed to the model.
4. The model evolved to incorporate some of the teachers' expectations.

Even with this collaboration, some misunderstandings were not resolved prior to the start of the project. This resulted in problems during the students' orientation.

Students were introduced to the project by the teachers during regular class periods. Teachers prepared overhead transparencies to explain the overall management plan to the students. This general orientation lasted over a period of a week, and during this time, students were kept in their regular classrooms while teachers explained the options available to each student individually. Initially, students were kept to the same pace. Self-pacing was introduced gradually to prevent the full weight of this responsibility from descending all at once. The overall student orientation process lasted well into topic 2 of the experiment, a period of about one month.

The starting dates and final dates for each topic were to be announced to the students before each topic. This

proved difficult at the outset because of a lack of experience with the materials. The dates were therefore not maintained rigidly. Time was generally allowed so that the slowest students would have at least two class periods between finishing one topic and starting the next.

The teachers decided to share the task of running the Teacher-taught class equally. This was not mandatory, and in some schools it might be desireable for one teacher to assume the full responsibility for this class. The teachers also decided to permit students to take tests only twice per week. This reduced the amount of supervision necessary and on the whole was a useful modification to the model.

2. THE INSTRUCTIONAL PLAN

Under this heading, student and teacher reactions related to the overall operating procedures and methods of instruction are reported.

The strongest criticism of the program by the students was that not enough time was available for proper completion of the units. Because of the importance of this criticism, its discussion is reserved for closer inspection in the next chapter.

Another strong criticism by the students was that teachers did not provide enough assistance for them. Typically, one of the following reasons was given:

(a) teachers did not spend enough time in the classroom, or (b) teachers would not answer questions but instead provided directions to consult student helpers or consult reference books. The latter complaint might be expected, since part of the treatment consisted of teaching students to be self-reliant. The former reason is discussed more fully in the next chapter, but part of this criticism resulted from teachers leaving their regular classrooms to work with groups of slow learners.

Teachers identified the main advantage of the experimental program as helping to locate slow learners and also helping to uncover their specific problems. Even with special help, the slow group experienced little success because of the large amount of reading and self-organization demanded of them. Teachers felt that a more manipulative approach would have been effective for these pupils, and that they needed a little time to 'goof-off'. The model provided few opportunities for Basic students to do enrichment work. During the first few topics, teachers observed that the slow students tended to skip over Activity Exercises which results of their Check-exercises indicated they should do; this was less marked during later topics. On the positive side, teachers felt that this group did not do less work than they would have done under a more traditional setting.

Besides being able to identify the slower students, teachers felt that the instructional plan helped to

distinguish pupils who needed constant supervision, those who were semi-independent, and those who were able to manage their learning for themselves.

With the better students, teachers were aware that at least two sets entered the Advanced level: those who liked mathematics and were mathematically oriented, and those who simply liked to achieve. This second group did not find all Advanced tasks attractive. They had difficulty with terminology and semantics, difficulty in verbalizing reasons and explanations, and dissapointment at receiving more objectives to complete. Teachers commented that

1. these students might stop trying for the Advanced group,

2. preconceived attitudes towards the learning of mathematics might be influencing these students,

3. coping with Advanced material might become easier as students were exposed to the tone of the sections from previous topics,

4. the skills required (analysis, synthesis, generalization) might take a long time to develop,

5. the students must understand that it is to their own advantage to be exposed to higher-level ideas, and

6. the value of Advanced work may have to be 'sold' to the students. (Subsequently this was done with moderate success. The main advantages pointed out to the students were recognition, higher marks, extended privileges, and the chal-

lenge of more involved mathematics.)

The Teacher-taught class was commented upon by many students. Of these, about two-thirds felt that it was useful and one-third that it was not. Most students attended this class at one time or another for specific help with certain ideas. About sixteen students were assigned to this class by a teacher at some time during the experiment. Many of these viewed this as a punishment because of the loss of freedom it imposed. Some of them stated that they refused to learn under those conditions. The provision to send 'problem' students to this class was effective administratively, but clearly led to hostility on the part of some pupils.

The original program called for both a Teacher-taught class and Mini-lectures. Mini-lectures were to be organized daily by some of the teachers, and entail assembling a small (five to ten) group of students having similar problems for a short (ten to fifteen minute) lecture. Instead, the Teacher-taught class came to be called the Mini-lecture. The teachers sessions with the slower students resembled the proposed Mini-lectures, but were not given an overt name. The planned Mini-lectures were to have included other levels of students as well. This misunderstanding was partly caused by a third form of special class; one that was to have been directed at students with self-management difficulties. This class was not overtly labelled by the experimenters, nor was it formally undertaken by the teachers. Improper com-

munication and a heavy work load on the teachers likely contributed to this miscue. These problems were recognized part way through the experiment, but no pertinent modifications were made.

Although teachers felt that students were doing no less work than under traditional methods, students generally felt they were doing no more. Most students indicated that they found Phase 2 useful, that they would most like to be in the Intermediate group, and that no peer censure attached to specific group membership. Teachers thought that most students wanted to attain or maintain a high (I or A) group status. Many students commented that they liked learning at their own speed and liked learning for themselves (although a few students made the contrary comment).

Because of the mobility of students, some teachers were apprehensive about the loss of 'class identity'. Teachers agreed that an extra teacher could well be used as a 'rover'; helping out wherever possible. This seems obvious in retrospect because of the complaints of students about lack of help and the chore of coping with the myriad student differences exposed by the program.

Most students had mixed opinions about the experimental program as a whole, but the majority felt it was better than the 'old' way of doing mathematics. They indicated that they did not think the procedures would work as well in other class subjects or even with other mathematics content. A few

selected subjective answer sheets included in Appendix V indicate the range of student reaction.

3. THE MATERIALS

In this section, reactions to the instructional materials are presented. The materials used included the Phase 1 and Phase 2 booklets, Non-routine Problems, Enrichment activities, prepared tests, filmstrips, and textbooks.

One or two students commented that the content covered in the course had already been done in grade six. This was partly true, but the content is dictated by the Department of Education curriculum. The ideas concerning rational numbers and decimals, although taken in grade six, are explored in greater depth in grade seven.

Questions 4 through 17 of the student questionnaire asked students to react to various aspects of the booklets they received. The frequency of positive responses would indicate that this part of the materials was well liked. An ordered list indicating the percentage of positive responses to the items is presented in Figure 5.

Many students commented that they liked marking their own Check-exercises, but that they found the Development confusing and too detailed. A few students complained that the Phase 1 booklets were too long (one complained they were too heavy), and covered too many ideas. Some Advanced

Question Number	Item Stem	Per Cent
9	The check-exercises	96.6
14	The solutions at the end of the topic	90.9
13	The review exercises	88.5
5	The objectives	84.5
10	The activity exercises in each section	81.1
8	The answers to questions in the development ..	81.0
17	The color-coding of the pages	80.3
15	The record page	80.2
6	The development in each section	78.4
7	The questions in the development	76.5
4	The introduction to each topic	73.6
11	The key ideas pages	72.6
12	The vocabulary list	63.9
16	The flowchart	61.9

Questions 4 through 17 of the student questionnaire asked students to rate a number of aspects of the materials. The above ordered list indicates how students responded. The value under Per Cent is the total of students indicating that the item was 'of some use' or 'very useful'.

FIGURE 5

SOME RESULTS FROM THE OBJECTIVE PORTION OF THE STUDENT QUESTIONNAIRE

students made similar comments about their Phase 2 booklets. Teachers complained about errors in the answers to the Check- and Activity Exercises, and also felt that for some sections for some students, the material might have been covered better verbally (in a regular classroom lecture situation).

A slip in nomenclature was made by the experimenters with respect to the booklets as well. The original model called for the buff pages in each section to contain both activities and exercises for students to use in improving their skills. Activities were later considered to be both too time consuming to devise and too difficult to administer. Therefore only exercises were included (with a few exceptions) and the name of these sections evolved from Activities and Exercises to Activity Exercises.

The Non-routine Problem section as implemented did not work out well. At the outset, this was to be an important part of the program, but teachers did not have time to supervise students engaged in this activity, nor to do the marking required. Teachers felt that this section was simply too difficult for most students. In reaction to these failings, two steps were taken. Students were encouraged to mark their own problems, and modifications were made to the record page to distinguish teacher-assessed from student-assessed results. Second, the name of the section was changed to 'Challengers', and teachers attempted to instil an attitude of challenge toward these problems in the students. These

modifications did not appear to be successful either. On the questionnaire, most students agreed that the Challengers were not too hard, but few felt they were useful. Teachers observed that students reported that they had satisfactorily solved the problems when in all likelihood they had not.

Less than half of the students tried the enrichment materials; of these about half reported that they enjoyed it. Some commented that it was not useful, some complained that they were never allowed to go there and one stated that it was always too crowded. Teachers felt that the Enrichment material was similar in level and tone to the Advanced material, and that only the Advanced students were making good use of it. They felt that materials of a more recreational nature were needed for the slower students, and especially for the Teacher-taught Class.

Although students did not feel that the tests were too hard, they did think there were too many tests and that they took too long to mark. Teachers felt that marking would be easier if the tests were converted, as far as possible, to multiple choice style. This was done for Topics 3 and 4.

Students enjoyed using the filmstrips, and teachers did not indicate that there were any problems in this area. The school provided six sets of different grade seven mathematics texts for the students. Whenever a student was un-unable to properly answer the last Check-exercise in a section, he was referred to a specific page in one of these

texts. Initially, students made good use of these reference books, but later in the program most preferred to go directly to a student helper or to the teacher for assistance. This situation likely developed because it was usually the poorer students who were directed to the texts. In retrospect, it is clear that the better students should have been directed to these reference books more than they were.

On the whole, most aspects of the materials were well liked and found useful by the students and teachers. Exceptions were the Non-routine Problems, the Enrichment, and the reference books. This would indicate not that these aspects should be removed from the program, but that more care should have been taken in their planning, construction, and implementation. It is also clear that the most pertinent addition in this area would be manipulative teaching aids.

4. CLASSROOM MANAGEMENT

This section is addressed to student and teacher reactions concerning classroom management implementations. Some management techniques were designed by the experimenters and some by the teachers.

Part of the experiment consisted of allowing a very free atmosphere to prevail; one in which students felt free to interact with each other. Student helpers were appointed from the better students. A number of pupils commented that

they obtained a great deal of help from their friends and that this was often better than getting help from a teacher. Teachers recognized that the student helpers were well accepted by most students and felt that they served a useful purpose.

A number of students complained that they were promised more freedom than they actually received. This is offset somewhat by other students' comment that many pupils 'fooled around' too much. Students also noticed (as did teachers) that some students merely copied answers from the back of the book and did not do their work. This diminished during later topics after these students were 'caught' on the tests.

A few students were annoyed because they felt the teacher took up too much time at the beginning of each class talking. Others complained that the booklets were not available for study just prior to the final tests. The reasons for these complaints are not known.

The teachers' main complaint in the management area concerned the increased paperwork involved with keeping the student records. They also felt that their role as teachers was substantially different during the experiment. (For a complete treatment of this topic, see te Kampe, 1970.) They felt more like guides and less like disciplinarians. They viewed their personal planning to be more directed towards individual students, and felt they did much more team

planning. They were able to give more attention to the slower students and felt that more was being accomplished in the classrooms. They saw a need for special skills as teachers: they had to cooperate closely with each other for moral support as well as sharing the work load, they had to be committed to the program, they relied heavily on their experience as teachers and grasp of teaching in general, and they had to help students adjust their expectations of the teachers in their new role. Although all teachers on the team had to work together, they felt that a smaller group in a smaller school could be just as effective.

5. FEASIBILITY SUMMARY

On the whole, teachers generally liked the experimental program. They agreed that although the model did not meet all individual differences, it had the potential for incorporating more alternatives. It made student differences more apparent and also created a situation in which the teachers could effectively deal with some of these differences.

The program was also generally liked by the students. Of the complaints made by them, most could have been resolved through more adequate planning. Some student complaints were directed at portions of the model which could not be changed; however, it is not clear that the system of instruction that is best liked by students would also provide the best learning

situation for them.

To the writer, there is no doubt that the described program of individualized instruction is a feasible alternative to existing methods. Ways in which this program could be improved, and areas where further study is needed are reported in the next chapter.

CHAPTER SIX

INTERPRETATIONS AND IMPLICATIONS

By outlining the reactions of the participants in the Hardisty experiment, the previous chapter has indicated the feasibility of the designed program. Areas of both success and inadequacy were made apparent. The present chapter focuses on these areas to explore the reasons for their occurrence and suggest alternate procedures which might be more effective. The feasibility report also reflects on the theory behind the procedures used. The theory presented in Chapter III is reviewed in light of the experimental observations. The chapter and this study are concluded with a general statement on the role of individualized instruction in secondary school mathematics.

1. COMMENTS ON IMPLEMENTATION

One point that cannot be overemphasized is the importance of a proper introduction of the program to the school. The three main groups effected by such experiments are the parents, the students, and the teachers. It is not uncommon for a project of the scope and duration of the Hardisty Experiment to draw some criticizm from parents. A child who is having difficulty with mathematics may find it

easy to blame the experimental program for his difficulties. To the knowledge of the experimenters, not a single case of such criticism was found. This must be at least partly attributed to the opportunities afforded to the parents to become acquainted with the experimental program at its outset.

The entire program was based on the students being able to acquire an understanding of the experimental procedures. The methods used for this goal have already been discussed. Part of the effectiveness of the program was due to the success of these methods.

The teachers were responsible for carrying out the orientation of both the students and parents. It is clear that their commitment and thorough understanding of the experiment was a fundamental necessity. Their commitment also enabled them to persevere under the role changes and heavy work load imposed on them. In the few cases where misunderstandings occurred, the cause could be traced to improper communication between the experimenters and the teachers.

2. COMMENTS ON OPERATION

As the experiment at Hardisty school progressed, areas of strength and weakness in the devised procedures became apparent. In this section, significant aspects of the operation are reported and reasons for failures and sug-

gestions for improvements are indicated.

The most important advantage of the program was that it identified certain groups of learners. As part of the instructional model, both the best and the poorest students (on the specific content) were separated from the main stream. This advantage was reinforced by three additional benefits of the program.

1. Teachers indicated that they were further able to separate these groups: the Basic into those who were poor in mathematics, and those who had difficulty in managing their their own studies; the Advanced into those who were naturally good at and liked mathematics, and those who wanted high achievement.

2. The program identified specific student problems. From looking at a student's flowchart, the teachers were able to locate relatively narrow areas of student weakness. Thus their time spent in helping individuals could be most efficiently used.

3. Teachers had more opportunity to help students individually. Because of the decrease in the amount of content preparation and class lecture time required of the teachers, they were able to spend a higher proportion of each class period attending to individuals. Basic students received extra help on fundamental concepts while Advanced students were provided with enriched learning experiences.

Clearly, the more homogeneous the group of

students, the easier it is for a teacher to provide instruction isomorphic to the students' needs. The most homegeneous grouping system would be achieved only when there were as many groups as there were students. However there is likely some method of division and number of groups which would make instruction most efficient for a particular curriculum content. The Hardisty Experiment indicated that three groups was not too many. But the question of whether distinguishing five groups (along the lines mentioned or some other criterion) would be feasible and provide better instruction remains unanswered.

There were four main areas in which the devised procedures were inadequate. These concerned the Non-routine Problems, the Activities and Exercises, the Enrichment, and the clerical tasks required of the teachers.

The reason for the failure of the Non-routine Problems (as outlined in the previous chapter) was that teachers did not adequately supervise this activity, nor did they evaluate students' attempted solutions. At the start of the program, the experimenters did not clearly indicate the importance of this section to the teachers, nor did they adequately outline the teachers' responsibilities in this area. When the difficulties became apparent at the end of the first topic, the teachers had already committed most of their time and were unwilling to accept the added responsibility this section entailed. If the teachers had been made

clearly aware of this task at the outset, it is likely that these difficulties would have been minimized. Further questions concerning the appropriateness, difficulty level, and instructional value of this section remain to be tested.

Although the exercises provided in the Activities and Exercises sections served a valid purpose, these sections could have been a great deal more useful if they had incorporated appropriate activities. The better students used these sections very little, as they were usually instructed to skip over them. However, the poor students who were having difficulty because of the reading demanded of them simply got 'more of the same' when they began the Activities and Exercises sections. The facility for incorporating laboratory or manipulative types of learning activities is readily available through this aspect of the plan. It would, in effect, provide an additional mode of instruction for students; and it would be used by those pupils most in need of this opportunity.

The enrichment material provided for students was effective in stabilizing their rates of progress. It provided meaningful activities for the faster students while the slower ones 'caught-up'. This is not a pedagogically defensible use of enrichment, because all students should have the opportunity to persue non-curriculum topics, or at least have respite from the daily routine of mathematics classes. More importantly, it is the slower students who would likely

take most profit from learning non-curriculum mathematics; for both the knowlege and the improvement of their attitude towards this subject. There are a number of problems: slow students usually have difficulty using their time effectively and therefore might get little out of enrichment, materials are difficult to devise for other than the better students, and it is usually only the better students who have the self-motivation to persue topics on their own. Under the present school system, where wide variation in student rate of progress is generally not allowed, these problems do not appear to be capable of solution.

Students criticized the teacher for being unavailable for help. This was caused partly by teachers taking groups of slow students to another room for special classes. However, the program did provide much unstructured time for the teachers and it was observed that in some cases they used a considerable amount of this in attending to administrative tasks. This was not desirable for the program and did not occur as often later in the experiment when teachers were more adapted to their new roles. The amount of clerical work necessary in marking the many exams and recording students' marks on both the teachers' and students' record pages was excessive. It was abated somewhat by the services of a teacher-aide, but still took the teachers away from duties more appropriate to the teaching of mathematics. Simply alleviating these many tasks would further the cause of

individualization by giving the teachers more time to spend with their students.

Computers are well suited to keeping track of this type of data. Since a large proportion of the tests given to students were constructed in a multiple choice style, answers reported on appropriate forms could be evaluated by the computer. Student records would then be updated automatically. Besides handling the clerical chores, the computer could also help monitor individual students' progress. By providing summary sheets on request and identifying which students could be grouped together for Mini-lectures, the computer would serve as a management assistant. Available information retrieval programs could also be used to access printed materials such as outlines for activities. This would make the problem of managing a large number of activities (in the Activities and Exercises sections) much easier for the teacher, and make the activities and materials more accessible to the students.

Both students and teachers found some parts of the Development in the materials too pedantic. Of course students always had the option of attending the Teacher-taught class, but it is likely that some content could be explained verbally more efficiently than through printed pages. This certainly occurred to the experimenters a number of times while they were preparing the Development sections. Using the same instructional model, perhaps some sections

should require a class lecture presentation to all students or to specific groups. This could be done only if the material was not sequentially dependent; as fast students would go on to following sections while teachers waited for a significant group to be ready for the class presentation.

The entire program was criticized by a number of people from the standpoint that the students did not cover as much of the curriculum as would have been covered using traditional methods. This was true, but must be viewed in proper perspective. First, an estimated three weeks of class time was lost through waiting for materials, the introduction of the new teaching method, and peripheral testing of the students. Secondly, a certain amount of 'overhead' time must be allowed for the installation of any new program. Over the course of a year, the time used by students and teachers to become aclimatized to the new approach would be less significant. Thirdly, the students clearly learned more about that portion of the curriculum that they did cover than a normally taught class would have. This is supported both by Sunde's thesis and the fact that teachers did not use students' phase 2 test results for determining the report card marks. Instead, they used the phase 1 test results. This kept the students' reported grades from being unacceptably high.

The described program was only implemented at one grade in one school. There does not appear to be any impediment to implementing the program in other schools, at other

grade levels, or in other curriculum subjects. The modifications found necessary by an experiment involving any combination of the above three shifts would be both valuable and interesting to educators. There is no doubt that much more than has been reported in this and the theses by Sunde and te Kampe can be examined concerning this model. It is almost certain that further ideas along these lines will be investigated. Suggestions for the direction of this investigation is the topic of the next section.

3. SUGGESTIONS FOR FURTHER RESEARCH

Following is a list of suggestions for further research. The items are ordered beginning with those areas that appear most fruitful to the writer. No comparative studies are indicated, partly because it is doubtful that significant improvement over traditional instruction methods could be demonstrated by this model over a short time span, and partly because the greatest need for research in this area is for development rather than formal evaluation.

1. The most significant single extension of this model would be the addition of appropriate activities to the Activities and Exercises sections. This would entail procurement of learning aids appropriate to the activities devised.

2. A computer should be added to the operation of the

model. Its uses could be to:

a) alleviate the clerical duties of teachers by marking tests and keeping track of student records,

b) help organize teachers' activities by providing summary sheets and signalling student problems,

c) provide direction for organizing Mini-lectures for both students and teachers, and

d) provide access to detailed instructions for activities for students and lesson plans for teachers.

3. A division of students into five groups should be tested; both the Basic and Advanced groups being subdivided along the criteria mentioned in the previous section. Criterion cutting points on Test 1 could be adjusted to distribute students appropriately among the three main groups, and evaluation by the teachers used to make the second divisions.

4. The content, purpose, administration, and justification of the Enrichment and Non-routine Problems sections should be reassessed; and suitable modifications made to incorporate these aspects into the model.

5. The model should be tested with different teachers, in different grades, with varying numbers of pupils, and with different subject matter.

4. COMMENTS ON THE THEORETICAL RATIONALE

Chapter III of this study presented four areas of theoretical background to the designed experimental procedures. The feasibility of the program indirectly indicates the adequacy of this theory.

Fourteen variables along which students' need to learn mathematics might be affected were detailed. It was asserted that these needs might be accounted for by providing a program of instruction which allowed: varying time on learning tasks, varying learning tasks, varying modes of presentation and enrichment opportunities.

On the whole, allowing students to proceed at their own pace did allow for the variables of intelligence, aptitude, study skills, motivation, reading skills, prior knowledge, attention span and physical condition. Allowing varying tasks also helped to individualize instruction along these variables. Since most students scored quite high on the tests at the end of each topic, it would appear that the program was successful in differentially assigning learning tasks to students. However, the Basic group usually proceeded most slowly indicating that reducing their number of learning tasks was not entirely successful in removing the time difference between high and low students. There remains room for compromise between these two implementations; but

it would appear that the required degree of mastery and the number of learning tasks for the high and low groups must be determined through trial and error testing with each segment of the curriculum. Expected variations in individuals ability would indicate that a dynamic method of determining this balance would be most appropriate.

The value of providing varying modes of presentation was not adequately tested by the experimental program, since only two well-defined possibilities were made operational. However, it is clear that the teacher-taught class choice was an asset to the total program in allowing for some student differences which otherwise would not have been met. Indications were that other modes would also be valuable, but the choice from alternatives would again depend upon student-curriculum centered trial and error testing. Dynamic choices from a set of established alternatives would, of course, be the best solution.

Enrichment opportunities were available in the program, but were not well used by the low group of students. There were two obvious reasons for this: the material was directed mainly towards the better students' interests, and the slower students seldom had the opportunity to engage in enrichment work. The choice of enrichment material seems to naturally tend towards the interests of the better students. In a program where the lower students were allowed an equitable proportion of enrichment time, a poor choice of material

for them could result in diminishing the usefulness of this extension to the curriculum.

Also emphasized was the need for a transparent organization of the instructional procedures. It was argued that if students understood the organizational structure of the program, they would be able to make some of their own learning decisions; if they made their own decisions, they would have to accept personal responsibility for their own learning; and if they accepted this responsibility then they would learn more efficiently. On question 40 of the student questionnaire, more than 70% of students agreed that they learned more mathematics this year than in other years. This lends credence to the above argument.

The concept of ability grouping has been in use by educators for some time. Although the devised program provided for only three groups, it did allow for frequent group changes and used achievement on similar content for group determination. A thorough study of group mobility and flexibility can be found in Sunde (1970). Class grouping was an integral part of the designed program and was certainly useful in determining learning tasks.

The final point emphasized was the need for and format of behavioral objectives. On the student questionnaire, more than 80% of students agreed that the supplied objectives were useful. This would indicate that the objectives were at least understandable to the students. Had they

been stated in a more formal way, it is unlikely that they would have been as useful. On question 36 of the questionnaire, more than 36% of students indicated that the objectives were useful in preparing for their tests. On question 37, more than 60% agreed that the tests were not too hard because: '... you know what all of the questions were going to be like.' This information would indicate that the format chosen was very effective for the experimental program.

On the whole, no information was collected from the experiment which would contradict the theoretical background to the project. Instead, questions were raised as to the further applicability of these ideas.

5. INDIVIDUALIZED INSTRUCTION IN SECONDARY SCHOOLS

It seems self-evident that the more that instruction is tailored to the individual needs of students, the more beneficial it will be for those students. An objection to this, that the random interaction provided in a school classroom is necessary to the child's total education, is vacuous. Individualization of instruction does not preclude such interactions, rather it must plan for them. The questions on how to arrange for the many experiences each child must have are not yet answered. But surely it is better to attempt the formulation of these answers than to continue to rely on the unorganized experience of each child

to determine the greatest part of his education.

The place to start appears to be with the construction of detailed educational objectives. If these objectives can be arranged in an ordered hierarchy, and if tests can be devised to assess mastery of these objectives, then the experiences any child needs should be capable of determination. This is a job which cannot be accomplished quickly, it may not even be capable of completion; but it is a job which can be started now.

Individualization may take one of two courses. If all students are expected to achieve the same objectives, it is logical that most graduates of the schools would be very similar in abilities. If the other course is taken, and different objectives are determined for different students, an obvious moral dilemma results. Yet it is this latter course which has been taken by default. The moral problem bothers no one because the experiences determining all learning are under little control. In effect, chance alone is allowed to determine which experiences an individual will have to prepare him for his environment.

At a time when the knowledge in the world is doubling every ten years, our educational system must be able to produce citizens of high ability in greater numbers than chance alone will provide. Methods such as the one described in this report seem the next logical step towards adapting education to the needs of our time.

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APPENDIX I

COST/TIME ACCOUNTING

Following is a statement of the money expended on the Hardisty Project. The initial grant was provided by the President's Humanities Research Fund.

<u>ASSETS</u>			
Initial Grant		$1500.00	
Received from Encyclopedia Britannica for Option to Print Materials		800.00	
Final Grant		200.00	$2500.00
<u>EXPENSES</u>			
Typing			
Topic 1	$ 82.00		
Topic 2	107.00		
Topic 3	134.00		
Topic 4	176.00	499.50	
Printing			
Topic 1	384.92		
Topic 2	536.89		
Topic 3	605.56		
Topic 4	337.93	1865.30	
Other			
Xeroxing	110.68		
Paper & Typewriter Ribbons	10.52		
Typewriter Rental	132.00	253.20	2618.00
NET DEFICIT			$ 118.00

The above statement does not include the services of three graduate student assistants. All were on assistanceships with the Department of Secondary Education (University of Alberta) and were allowed half of their time to work on the project. The expense to the University in this regard would be an additional $4500.00, which resulted in six hours

of work per week by three people for thirty weeks, or four hundred and fifty man-hours.

Also not included are the services of both Mortlock and Sigurdson. It can be estimated that these two people contributed an additional four hundred man-hours to the project.

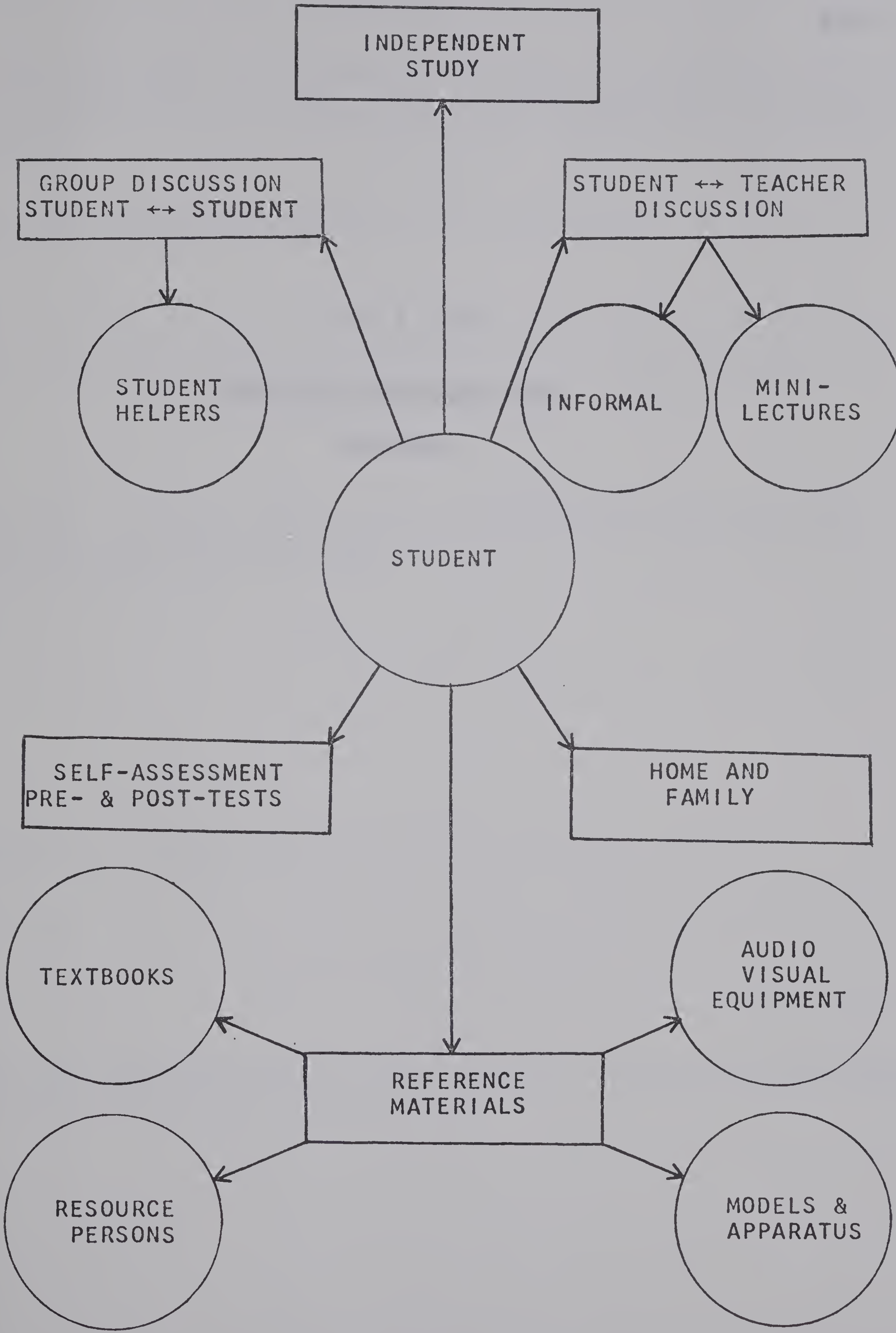
As a result of these expenditures, about three hundred of the four units of grade seven mathematics material were produced. Each set consisted of over one thousand typewritten pages and would occupy an average student for about three and one-half months. Also produced were three masters theses and a wealth of experience for the graduate students and the teachers at Hardisty school.

APPENDIX II

HANDOUTS TO THE PARENTS AND PRE-EXPERIMENT ACTIVITIES

Included in this appendix are two items prepared by the teachers at Hardisty school. Both were made available to parents. The first is a diagram showing the student as the hub of resources available for his education.

During the first half of the school year, just prior to the experiment, teachers at Hardisty had designed and implemented their own individualization program. The nature of their efforts is clarified by the second item in this appendix. The aspects felt to be important by the teachers and the nature of the materials they used is illustrated by the 'H.I.P.' paper.


INDEPENDENT STUDY
GROUP DISCUSSION STUDENT ↔ STUDENT
STUDENT ↔ TEACHER DISCUSSION
STUDENT HELPERS
INFORMAL
MINI-LECTURES
STUDENT
SELF-ASSESSMENT PRE- & POST-TESTS
HOME AND FAMILY
TEXTBOOKS
AUDIO VISUAL EQUIPMENT
REFERENCE MATERIALS
RESOURCE PERSONS
MODELS & APPARATUS

H. I. P.

Hardisty Individualized

Progress

PRE-TEST

These questions have been designed to allow you to evaluate your own knowledge of the Individualized Progress mathematics program in our school.

1. How many students participate in the program?

2. How many periods per week does your child study mathematics?

3. Describe the objectives of the Individualized Progress math program. Be sure to include such terms as student responsibility, multi-faceted approach, etc..

4. Outline the proper use of pre-tests and post-tests?

5. What are the advantages of this program of Individualized Progress over more traditional approaches?

If you score 100% on this test you should be up here and I should be sitting in your seat.
If you did not do well please be patient and listen carefully.

Overview:

It is the belief of the math teachers at Hardisty Junior High School that the most effective learning will occur through a variety of teaching methods and learning activities. That such a variety of methods and activities be utilized is readily seen when it is realized that learners and what is to be learned vary to a great degree. No two people are alike nor do they necessarily learn in the same way. It is then the responsibility of the school to provide maximum opportunity for each learner to proceed at his own speed; to cover as much material as he can handle; to choose the method of learning that best suits him; and to select those aids to learning which makes understanding the simplest.

The math department at our school has attempted to meet these responsibilities with a program of Individualized Progress. It is our purpose now to explain this program to you. We will attempt to do this by providing this information using the same format as the lessons which serve as one avenue of learning for your youngsters.

Objectives	Development
I - A To acquaint parents with the times and location of when this program is in operation.	Location - upper floor on the East wing. Times - One period each day, except Wednesday (for times consult your child's timetable)
I-B To acquaint parents with the operational procedures of the program.	There are 9 Grade VII classes in the school. These 9 classes are divided into two sections: One section consists of 5 classes and the other of 4. Each section takes math at the same time.

Objectives	Development
I - C To acquaint parents with the overall objectives of our math program.	Objectives of our math program. 1. To provide the opportunity for individual and continuous progress. 2. To provide students with as many avenues or methods of learning a given concept as possible. 3. To provide a means by which the teacher gains more individual contact with the student. 4. To provide the student with the means for realistic self-evaluation. 5. To provide students with objectives that are specific and measurable. 6. To provide enrichment for maximum growth of the learner. 7. To allow teams of teachers to pool talents, capabilities, share tasks, information, and responsibilities.

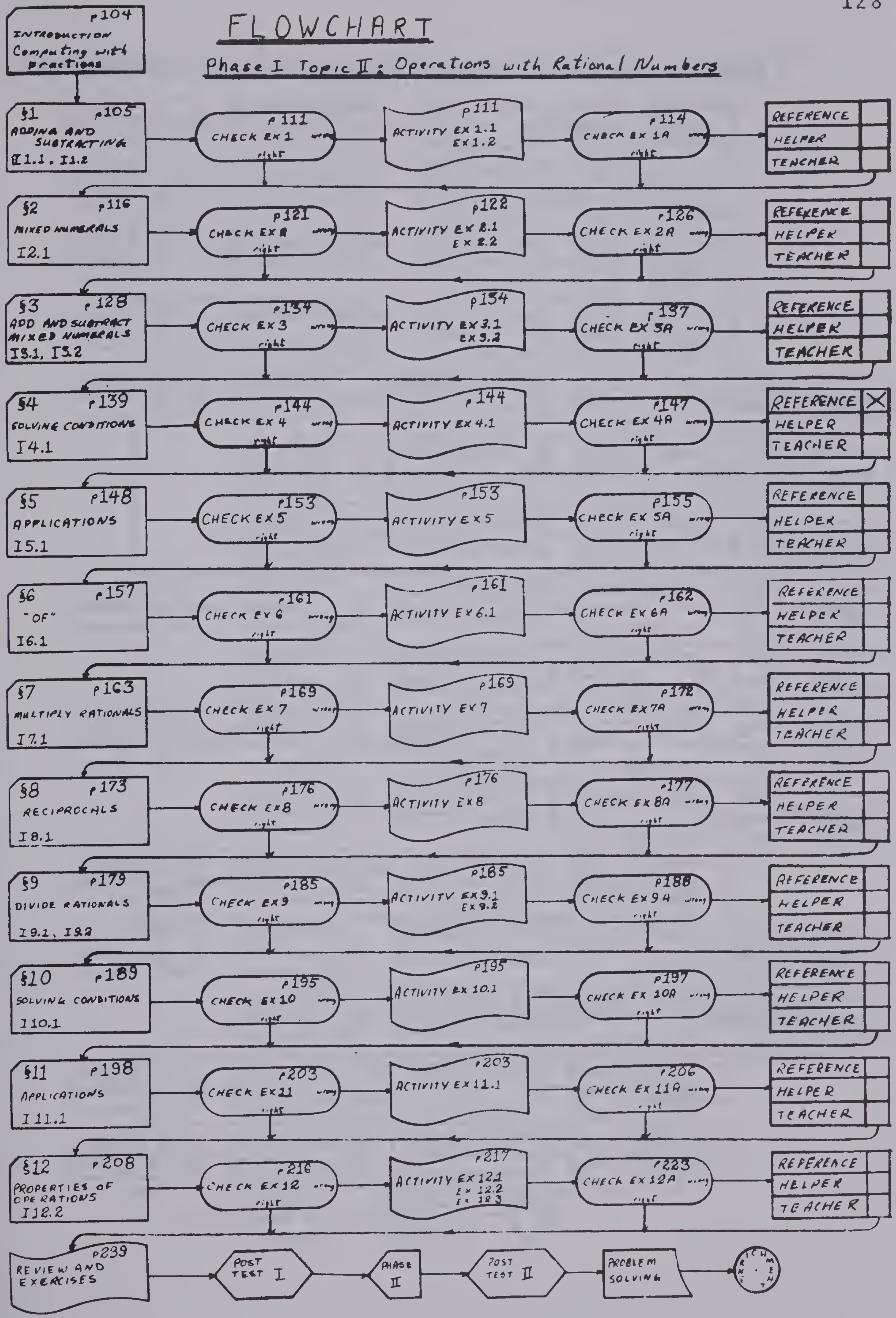
Objectives	Development
	8. To invite parents to visit the school, to criticize the program and to work with teachers to provide the best possible education for <u>their</u> children and <u>our</u> students.
I - D To invite parents to ask questions.	An attempt will be made to answer all questions.
I - E To invite parents to meet the other teachers involved in the program.	Each parent is invited to follow the teacher of his or her child to their classrooms.
I - F To thank parents for coming.	Remember that you are welcome at any time.

POST TEST

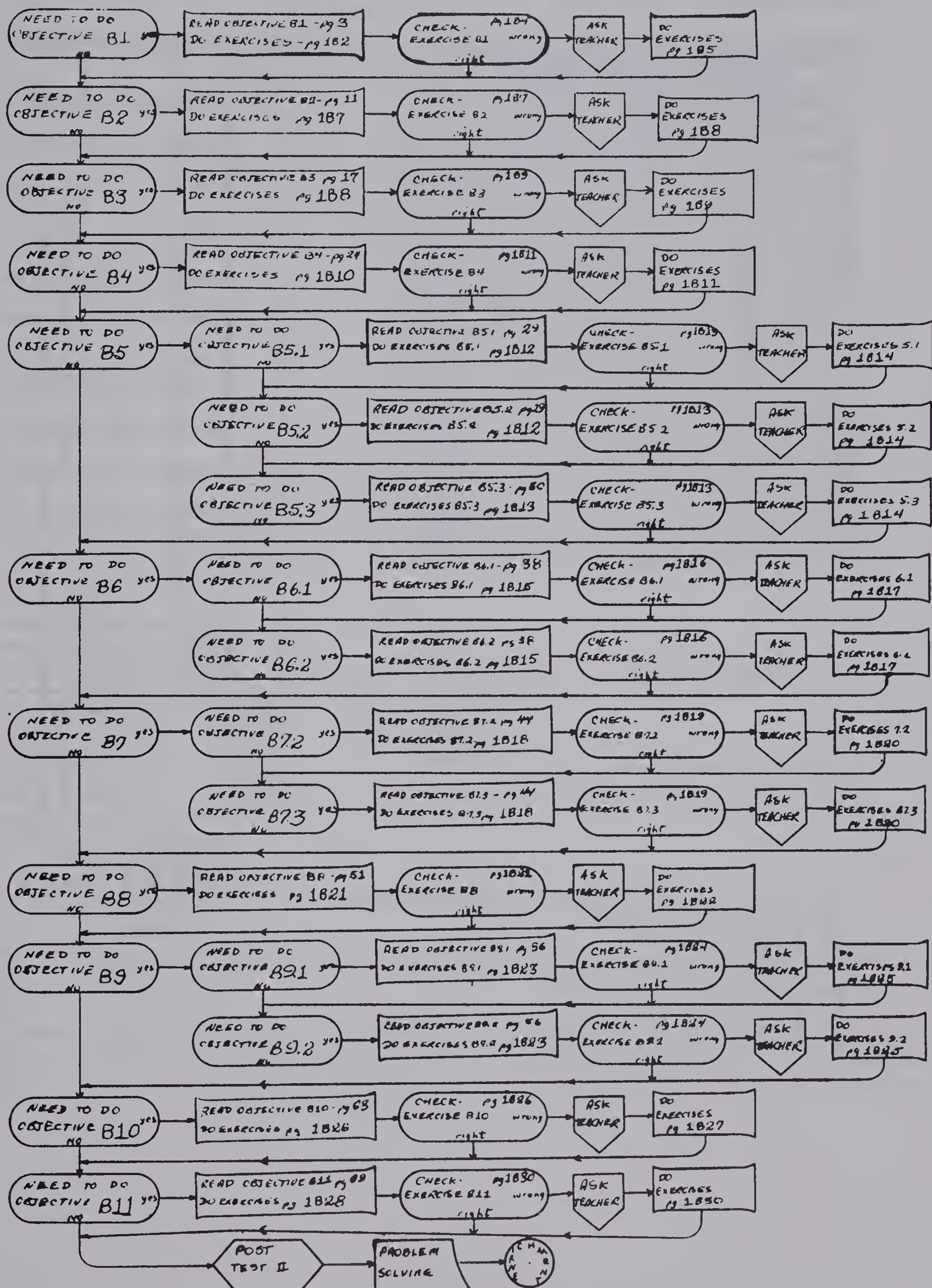
Re-write the pre-test.
If your score is less than 100% and if you are not totally convinced that our program of Individualized Progress will provide the best possible mathematics education for your child then ask for further explanation and clarification.

APPENDIX III

SAMPLES OF THE MATERIALS


FLOWCHART
Phase I Topic II: Operations with Rational Numbers
p104 INTRODUCTION Computing with Fractions
§1 p105 ADDING AND SUBTRACTING I1.1, I1.2
CHECK EX 1 p111 wrong right
ACTIVITY EX 1.1 EX 1.2 p111
CHECK EX 1A p114 wrong right
REFERENCE
HELPER
TEACHER
§2 p116 MIXED NUMERALS I2.1
CHECK EX 2 p121
ACTIVITY EX 2.1 EX 2.2 p122
CHECK EX 2A p126
§3 p128 ADD AND SUBTRACT MIXED NUMERALS I3.1, I3.2
CHECK EX 3 p134
ACTIVITY EX 3.1 EX 3.2 p134
CHECK EX 3A p137
§4 p139 SOLVING CONDITIONS I4.1
CHECK EX 4 p144
ACTIVITY EX 4.1 p144
CHECK EX 4A p147
§5 p148 APPLICATIONS I5.1
CHECK EX 5 p153
ACTIVITY EX 5 p153
CHECK EX 5A p155
§6 p157 "OF" I6.1
CHECK EX 6 p161
ACTIVITY EX 6.1 p161
CHECK EX 6A p162
§7 p163 MULTIPLY RATIONALS I7.1
CHECK EX 7 p169
ACTIVITY EX 7 p169
CHECK EX 7A p172
§8 p173 RECIPROCALS I8.1
CHECK EX 8 p176
ACTIVITY EX 8 p176
CHECK EX 8A p177
§9 p179 DIVIDE RATIONALS I9.1, I9.2
CHECK EX 9 p185
ACTIVITY EX 9.1 EX 9.2 p185
CHECK EX 9A p188
§10 p189 SOLVING CONDITIONS I10.1
CHECK EX 10 p195
ACTIVITY EX 10.1 p195
CHECK EX 10A p197
§11 p198 APPLICATIONS I11.1
CHECK EX 11 p203
ACTIVITY EX 11.1 p203
CHECK EX 11A p206
§12 p208 PROPERTIES OF OPERATIONS I12.2
CHECK EX 12 p216
ACTIVITY EX 12.1 EX 12.2 EX 12.3 p217
CHECK EX 12A p223
p239 REVIEW AND EXERCISES
POST TEST I
PHASE II
POST TEST II
PROBLEM SOLVING
ENRICHMENT

FLOWCHART: PHASE II (BASIC) TOPIC I: RATIONAL NUMBERS AND FRACTIONS



TOPIC 2 RECORD PAGE

NAME ____________________

CLASS ____________________

OBJECTIVES	B	I	A
B 1.1			
I 1.1		—	
B 1.2			
I 1.2		—	
B 2.1			
I 2.1		—	
B 3.1			
I 3.1		—	
B 3.2			
I 3.2		—	
B 4.1			
I 4.1		—	
B 5.1			
I 5.1		—	
I 6.1		—	
B 7.1			
I 7.1		—	
B 8.1			
I 8.1		—	
B 9.1			
I 9.1		—	
I 9.2		—	
B 10.1			
I 10.1		—	
B 11.1			
I 11.1		—	
I 12.1		—	
I 12.2		—	
I 12.3		—	
A 1			—
A 2			—
A 3			—
A 4			—
A 5			—
A 6			—
PHASE I Sub Total			
Total			
Possible	0	17	
	17		

CHALLENGERS (Non-Routine Problem Solving)

Problem Number	Attempted This Problem	Success	
		Student Assessed	Teacher Assessed
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			

≤ 7 BASIC

8 to 13 INTERMEDIATE

≥ 14 ADVANCED

PHASE II GROUP ______

COMMENT: ____________________

____

PHASE II B Group		PHASE I & II Total	
		Possible Total	12
PHASE II I Group		PHASE I & II Sub Total	
		Total	
PHASE II A Group		PHASE I & II Sub Total	

TOPIC 2

PHASE I

OPERATIONS WITH RATIONAL NUMBERS

OPERATIONS WITH RATIONAL NUMBERS

INTRODUCTION

In almost every walk of life, at work or at home, people daily have to add, subtract, multiply or divide rational numbers; i.e. they have to use one or more of the four basic operations with rational numbers.

A girl may want to make a recipe for $\frac{2}{3}$ as many people as the directions on the packet indicate or may want to know the cost of $1\frac{3}{4}$ yards of material. A boy may want to know how much he would earn if he increased his paper round by $\frac{1}{3}$ or may want to know the combined length of a piece $1\frac{3}{4}$" long and one $2\frac{5}{8}$" long in a model he is planning.

Operations with rational numbers will also be used often in the work you will be doing in mathematics from now until you finish school. Since you will be using these ideas so much, it is important that you be able to use them accurately and quite quickly.

The operations that you will be studying in this topic are those listed above - addition, subtraction, multiplication and division. You have already met them before.

We will also be concerned with mixed numerals (eg. $4\frac{2}{5}$), with solving conditions involving rational numbers (eg. $n + \frac{2}{3} = \frac{3}{4}$), with reciprocals, (eg. $\frac{2}{3}$ and $\frac{3}{2}$), with properties of rational numbers (eg. commutative property) and with applying rational numbers to answer questions about everyday situations.

OBJECTIVE B1.1

To find the sum of two or more rational numbers named by fractions.

Example

Find the sums and write each as a basic fraction.

A. $\frac{3}{4} + \frac{5}{8} + \frac{1}{4}$

B.
$$\begin{array}{r} \frac{1}{3} \\ \frac{2}{5} \\ + \frac{5}{3} \\ \hline \end{array}$$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A. $\frac{3}{4} + \frac{5}{8} + \frac{1}{4} = \frac{6}{8} + \frac{5}{8} + \frac{2}{8}$

$= \frac{13}{8}$

B.
$$\begin{array}{rcl} \frac{1}{3} & = & \frac{5}{15} \\ \frac{2}{5} & = & \frac{6}{15} \\ + \frac{5}{3} & = & \frac{25}{15} \\ \hline & & \frac{36}{15} = \frac{36 \div 3}{15 \div 3} = \frac{12}{5} \end{array}$$

OBJECTIVE I1.1

To find the sum of two or more rational numbers named by fractions.

Example

Find the sums and write each as a basic fraction.

A. $\frac{1}{5} + \frac{1}{4} + \frac{5}{6}$

B.
$$\begin{array}{r} \frac{1}{9} \\ \frac{7}{6} \\ + \frac{1}{4} \\ \hline \end{array}$$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A. $\frac{1}{5} + \frac{1}{4} + \frac{5}{6} = \frac{12}{60} + \frac{15}{60} + \frac{50}{60}$

$= \frac{77}{60}$

B.
$$\begin{array}{rcl} \frac{1}{9} & = & \frac{4}{36} \\ \frac{7}{6} & = & \frac{42}{36} \\ + \frac{1}{4} & = & \frac{9}{36} \\ \hline & & \frac{55}{} \end{array}$$

OBJECTIVE B1.2

To find the difference of two rational numbers named by fractions.

Example

Find the differences and write each as a basic fraction.

A. $\frac{7}{3} - \frac{5}{4}$

B. $\begin{array}{r} \frac{8}{9} \\ - \frac{1}{3} \\ \hline \end{array}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A. $\frac{7}{3} - \frac{5}{4} = \frac{28}{12} - \frac{15}{12}$
$= \frac{13}{12}$

B. $\begin{array}{rcr} \frac{8}{9} & = & \frac{8}{9} \\ - \frac{1}{3} & = & \frac{3}{9} \\ \hline & & \frac{5}{9} \end{array}$

OBJECTIVE I1.2

To find the difference of two rational numbers named by fractions.

Example

Find the differences and write each as a basic fraction.

A. $\frac{7}{8} - \frac{1}{5}$

B. $\begin{array}{r} \frac{17}{5} \\ - \frac{5}{6} \\ \hline \end{array}$

Criterion: Method correct in each example and no more than one error in computation.

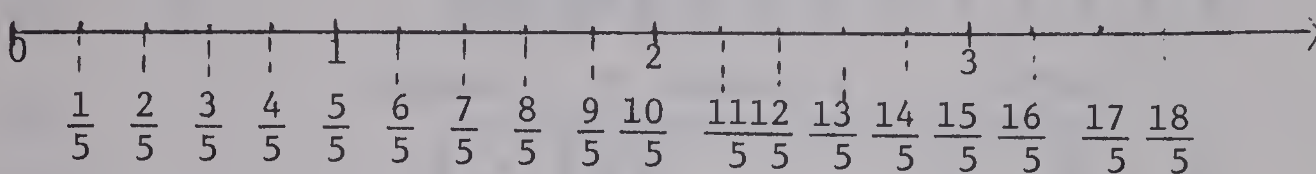
SOLUTION

A. $\frac{7}{8} - \frac{1}{5} = \frac{35}{40} - \frac{8}{40}$
$= \frac{27}{40}$

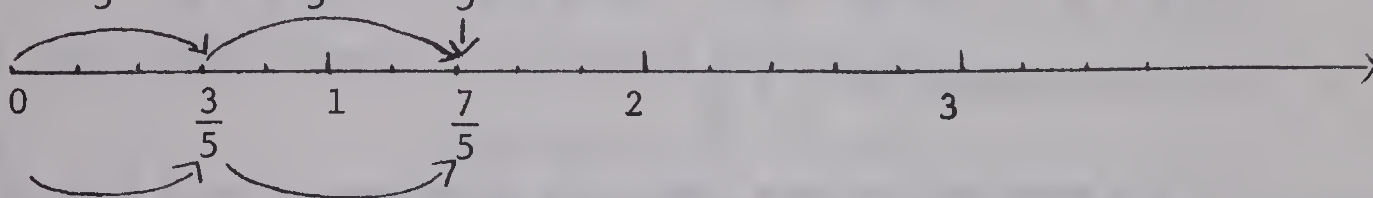
B. $\begin{array}{rcr} \frac{17}{5} & = & \frac{102}{30} \\ - \frac{5}{6} & = & \frac{25}{30} \\ \hline & & \frac{77}{30} \end{array}$

SECTION 1. Operations of addition and subtraction with rational numbers

Suppose we want to add $\frac{3}{5}$ and $\frac{4}{5}$. We can show this on a numberline subdivided into fifths.



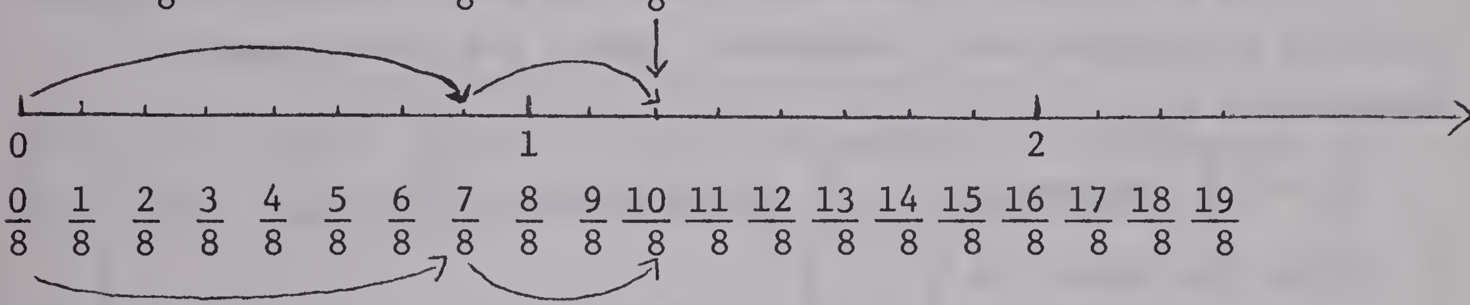
$$\frac{3}{5} + \frac{4}{5} = \frac{7}{5}$$



3 fifths + 4 fifths = 7 fifths

Here is another example, this time with eights. $\frac{7}{8} + \frac{3}{8}$

$$\frac{7}{8} + \frac{3}{8} = \frac{10}{8}$$



7 eights + 3 eights = 10 eights

The sum is $\frac{10}{8}$. Since answers are usually given as basic fractions we reduce $\frac{10}{8}$ to its basic fraction.

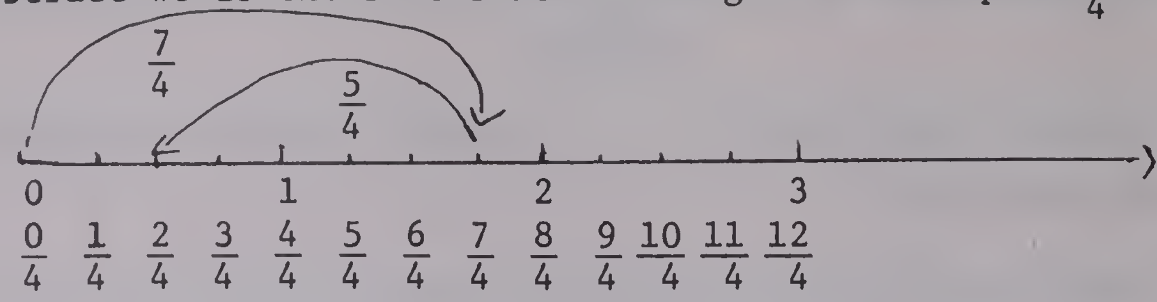
$$\frac{10}{8} = \frac{10 \div 2}{8 \div 2} = \frac{5}{4}$$

1. Use the numberlines above (if necessary) to find

(1) $\frac{4}{5} + \frac{6}{5}$ (2) $\frac{3}{8} + \frac{9}{8}$

Answers: 1. (1) $\frac{10}{5} = \frac{2}{1} = 2$ (2) $\frac{12}{8} = \frac{3}{2}$

To subtract we do the same sort of thing. For example $\frac{7}{4} - \frac{5}{4}$



7 fourths − 5 fourths = 2 fourths

$$\frac{7}{4} - \frac{5}{4} = \frac{2}{4} = \frac{2 \div 2}{4 \div 4} = \frac{1}{2}$$

2. Write answers to the following in your work book.

(1) $\frac{7}{8} - \frac{3}{8}$ (2) $\frac{11}{16} - \frac{5}{16}$ (3) $\frac{2}{3} + \frac{4}{3}$ (4) $\frac{7}{4} - \frac{7}{4}$

3. Write the answer to $\frac{7}{4} + \frac{1}{2}$

In the examples we have done, the denominators have been the same in both fractions.

What about $\frac{5}{8} + \frac{1}{2}$?

In order to add them the denominators must be the same.

We can use common denominators and the least common denominator (L.C.D.) is usually most convenient. Here, the least common denominator is 8.

$\frac{5}{8} + \frac{1}{2}$ becomes $\frac{5}{8} + \frac{4}{8}$ and this sum is $\frac{9}{8}$.

4. Write the answer to $\frac{2}{3} + \frac{3}{4}$

Here is another example $\frac{2}{5} + \frac{5}{4}$

The L.C.D. is 20

$$\frac{2}{5} = \frac{2 \times 4}{5 \times 4} = \frac{8}{20} \quad \text{and} \quad \frac{5}{4} = \frac{5 \times 5}{4 \times 5} = \frac{25}{20}$$

Thus $\frac{2}{5} + \frac{5}{4} = \frac{8}{20} + \frac{25}{20}$

$= \frac{33}{20}$ (and this is a basic fraction)

Answers: 2: (1) $\frac{4}{8} = \frac{1}{2}$ (2) $\frac{6}{16} = \frac{3}{8}$ (3) $\frac{6}{3} = \frac{2}{1}$ or 2 (4) $\frac{0}{4}$ or 0

3: $\frac{7}{4} + \frac{1}{2} = \frac{7}{4} + \frac{1 \times 2}{2 \times 2} = \frac{7}{4} + \frac{2}{4} = \frac{9}{4}$ 4: $\frac{2}{3} + \frac{3}{4} = \frac{2 \times 4}{3 \times 4} + \frac{3 \times 3}{4 \times 3} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$

Yet another example:

$$\frac{5}{6} + \frac{9}{10}$$ Find the L.C.D. It is 30.

$$= \frac{5 \times 5}{6 \times 5} + \frac{9 \times 3}{10 \times 3}$$ Find equivalent fractions with this denominator.

$$= \frac{25}{30} + \frac{27}{30}$$ Add the numerators.

$$= \frac{52}{30}$$ Is the answer a basic fraction?

$$= \frac{52 \div 2}{30 \div 2}$$ If not, reduce it to a basic fraction.

$$= \frac{26}{15}$$

Now a subtraction example. The steps are similar.

$$\frac{9}{8} - \frac{5}{6}$$ Find the L.C.D. It is 24

$$= \frac{9 \times 3}{8 \times 3} - \frac{5 \times 4}{6 \times 4}$$ Find equivalent fractions with this denominator. Find the difference of the numerators.

$$= \frac{27}{24} - \frac{20}{24}$$ The answer is a basic fraction. We are finished.

$$= \frac{7}{24}$$

Now look at this. Suppose in the last example we had used 48 as the common denominator instead of 24. The steps would be:

$$\frac{9}{8} - \frac{5}{6}$$

$$= \frac{9 \times 6}{8 \times 6} - \frac{5 \times 8}{6 \times 8}$$

$$= \frac{54}{48} - \frac{40}{48}$$

$$= \frac{14}{48}$$ The result here is not a basic fraction. However, reducing it we get the same as above.

$$= \frac{14 \div 2}{48 \div 2}$$

$$= \frac{7}{24}$$

We can use any common denominator when adding or subtracting rational numbers. However, the least common denominator usually gives the result with the least amount of work.

5. Find $\frac{2}{3} + \frac{3}{5} + \frac{7}{4}$

To find the sum of three or more rational numbers, you find the L.C.D. of all the denominators, find equivalent fractions with this denominator and then add the numerators.

For example:

Find

$$\begin{array}{rcccc} \frac{5}{6} & = & \frac{5 \times 6}{6 \times 6} & = & \frac{30}{36} \\ \frac{5}{4} & = & \frac{5 \times 9}{4 \times 9} & = & \frac{45}{36} \\ +\ \frac{5}{9} & = & \frac{5 \times 4}{9 \times 4} & = & \frac{20}{36} \\ \hline & & & & \frac{95}{36} \end{array}$$

The L.C.D. of 6, 4, and 9 was 36.

Now read OBJECTIVES I1.1 and I1.2 and their examples. These tell you what you are expected to be able to do for this section. When ready turn to and do CHECK EXERCISES 1.1 and 1.2.

Answers: 5: $\frac{2}{3} + \frac{3}{5} + \frac{7}{4} = \frac{2 \times 20}{3 \times 20} + \frac{3 \times 12}{5 \times 12} + \frac{7 \times 15}{4 \times 15} =$

$\frac{40}{60} + \frac{36}{60} + \frac{105}{60} = \frac{181}{60}$

CHECK EXERCISE 1

I1.1 Add the following and write each sum as a basic fraction.

a) $\frac{6}{25} + \frac{3}{10} + \frac{4}{5}$

b) $\frac{5}{8} + \frac{11}{16} + \frac{2}{3}$

c) $\frac{3}{4} + \frac{11}{15} + \frac{7}{12}$

d) $\begin{array}{r} \frac{9}{10} \\ \frac{3}{4} \\ + \frac{2}{5} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{9}{10} \\ \frac{11}{12} \\ + \frac{3}{4} \\ \hline \end{array}$

I1.2 Find the differences and write each as a basic fraction.

a) $\frac{2}{3} - \frac{1}{6}$

b) $\frac{4}{5} - \frac{1}{2}$

c) $\frac{4}{9} - \frac{1}{6}$

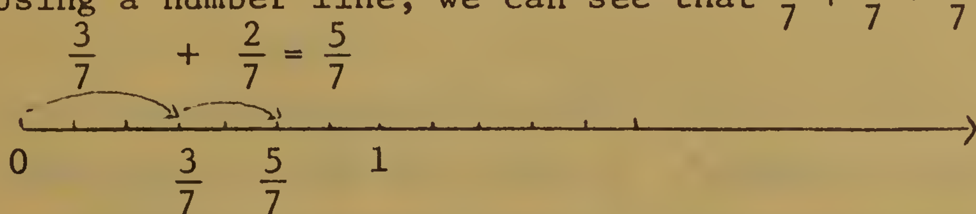
d) $\frac{7}{8} - \frac{1}{5}$

e) $\frac{10}{6} - \frac{3}{4}$

- Check your answers with those given at the end of the topic.
- If you are not certain how to add rational numbers, or if you had more than one part incorrect in CHECK EXERCISE 1.1; read section 1 carefully and do activity exercises 1.1.
- If you are not certain how to subtract rational numbers, or if you had more than one part incorrect in CHECK EXERCISE 1.2; read section 1 carefully and do activity exercises 1.2.
- Otherwise, go on to section 2.

Activity Exercises 1.1

1. Using a number line, we can see that $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$



a) Write the fraction that equals $\frac{2}{7} + \frac{4}{7}$.

b) Add the following. Use a number line if you wish, but try to do them without.

i) $\frac{4}{9} + \frac{3}{9}$ ii) $\frac{5}{4} + \frac{2}{4}$ iii) $\frac{3}{10} + \frac{8}{10}$ iv) $\frac{1}{5} + \frac{3}{5}$

$\frac{3}{8} + \frac{1}{4}$ cannot be added as easily because they have different denominators.

But $\frac{1}{4} = \frac{1 \times 2}{4 \times 2} = \frac{2}{8}$. Now we have $\frac{3}{8} + \frac{2}{8} = \frac{5}{8}$.

Study the following examples:

a) $\frac{3}{4} + \frac{2}{5}$

$= \frac{3 \times 5}{4 \times 5} + \frac{2 \times 4}{5 \times 4}$ L.C.D. is 20

$= \frac{15}{20} + \frac{9}{20}$

$= \frac{23}{20}$

b) $\frac{1}{4} + \frac{5}{6}$

$= \frac{1 \times 3}{4 \times 3} + \frac{5 \times 2}{6 \times 2}$ L.C.D. is 12

$= \frac{3}{12} + \frac{10}{12}$

$= \frac{13}{12}$

Note: We must have common denominators in order to add rational numbers. The least common denominator is usually most convenient.

c) Add the following. Be sure you have a common denominator.

i) $\frac{1}{2} + \frac{3}{4}$

ii) $\frac{2}{3} + \frac{1}{6}$

iii) $\frac{8}{5} + \frac{4}{3}$

iv) $\frac{4}{7} + \frac{5}{9}$

v) $\frac{11}{10} + \frac{11}{7}$

Adding $\frac{3}{10} + \frac{3}{10}$ we get $\frac{6}{10}$. But $\frac{6}{10}$ can be reduced to a basic fraction of $\frac{3}{5}$. ($\frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$).

Note: We always reduce a fraction to its basic fraction before considering the question finished.

d) Add the following. Be sure they are reduced to basic fractions when necessary.

i) $\frac{5}{12} + \frac{1}{4}$

ii) $\frac{1}{15} + \frac{5}{6}$

iii) $\frac{3}{10} + \frac{7}{8}$

Adding three or more rational numbers is done just as for adding two rational numbers. Be sure they ALL are expressed with common denominators and the final answers are BASIC FRACTIONS.

$\frac{2}{3} + \frac{4}{5} + \frac{5}{6}$

$= \frac{20}{30} + \frac{24}{30} + \frac{25}{30}$ – common denominators (L.C.M. is 30)

$= \frac{69}{30}$

$= \frac{23}{10}$ – basic fraction

We can also use column form when adding 2 or more fractions.

$$\begin{array}{rcrl} & \frac{3}{8} & = & \frac{9}{24} \\ & \frac{5}{6} & & \frac{20}{24} \qquad \text{L.C.D. is 24} \\ + & \frac{2}{3} & + & \frac{16}{24} \\ \hline & & & \frac{45}{24} \\ & & = & \frac{15}{8} \qquad \text{basic fraction} \end{array}$$

e) Add the following and express the sums as basic fractions.

i) $\frac{5}{6} + \frac{1}{2} + \frac{2}{3}$

ii) $\frac{1}{2} + \frac{3}{8} + \frac{1}{4}$

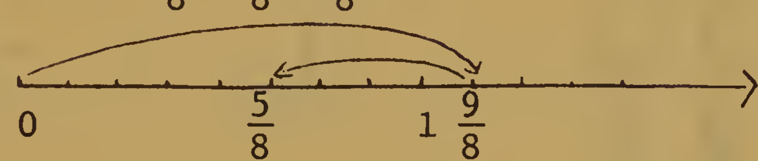
iii) $\frac{3}{8} + \frac{5}{12} + \frac{8}{15}$

iv) $\begin{array}{rr} & \frac{2}{3} \\ & \frac{2}{5} \\ + & \frac{7}{15} \\ \hline \end{array}$

v) $\begin{array}{rr} & \frac{2}{3} \\ & \frac{3}{4} \\ + & \frac{5}{6} \\ \hline \end{array}$

Activity Exercises 1.2

We can also use the number line to find the difference between two rational numbers. $\frac{9}{8} - \frac{4}{8} = \frac{5}{8}$



a) Subtract the following. Use a number line if necessary, but try to do it without.

i) $\frac{5}{9} - \frac{4}{9}$ ii) $\frac{12}{5} - \frac{8}{5}$ iii) $\frac{7}{8} - \frac{5}{8}$

In order to subtract $\frac{4}{5}$ from $\frac{9}{10}$ we must express both fractions with a common denominator.

$$\frac{9}{10} - \frac{4}{5}$$

$$= \frac{9}{10} - \frac{8}{10} \qquad \left(\frac{4}{5} = \frac{4 \times 2}{5 \times 2} = \frac{8}{10} \right)$$

$$= \frac{1}{10}$$

b) Subtract the following. Use the least common denominator.

i) $\frac{9}{5} - \frac{2}{15}$ ii) $\frac{5}{8} - \frac{2}{6}$ iii) $\frac{5}{7} - \frac{2}{5}$

In the following example we find we get an answer of $\frac{2}{12}$.

$$\frac{7}{12} - \frac{5}{12} = \frac{2}{12}$$

But, we do not leave the answer as $\frac{2}{12}$. We reduce it to its BASIC FRACTION.

$$\frac{2}{12} = \frac{2 \div 2}{12 \div 2} = \frac{1}{6}$$

Thus $\frac{7}{12} - \frac{5}{12} = \frac{2}{12}$

$$= \frac{1}{6}$$

We can also subtract in column form.

$$\begin{array}{rcr} \frac{4}{5} & = & \frac{4}{30} \\ -\ \frac{1}{10} & & -\ \frac{3}{30} \\ \hline & & \frac{1}{30} \end{array}$$

NOTE: ALWAYS EXPRESS THE ANSWER AS A BASIC FRACTION.

c) Find the differences and express each as a basic fraction.

i) $\frac{7}{16} - \frac{1}{4}$

ii) $\frac{3}{10} - \frac{1}{20}$

iii) $\frac{9}{20} - \frac{1}{5}$

iv) $\begin{array}{r} \frac{3}{4} \\ -\ \frac{2}{3} \\ \hline \end{array}$

v) $\begin{array}{r} \frac{9}{5} \\ -\ \frac{4}{7} \\ \hline \end{array}$

- Check your answers with those given at the end of the topic.
- Read objective I1.1 and I2.1.
- If you feel you are ready, do CHECK EXERCISE 1A.

CHECK EXERCISE 1A

I1.1 Add the following and write each sum as a basic fraction.

a) $\frac{1}{4} + \frac{2}{9} + \frac{5}{12}$

b) $\frac{1}{4} + \frac{4}{25} + \frac{3}{10}$

c) $\frac{3}{14} + \frac{4}{21} + \frac{5}{28}$

d) $\begin{array}{r} \frac{1}{6} \\ \frac{2}{3} \\ +\ \frac{0}{5} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{3}{8} \\ \frac{1}{6} \\ +\ \frac{5}{12} \\ \hline \end{array}$

11.2 Find the differences and write each as a basic fraction.

a) $\frac{7}{12} - \frac{1}{4}$

b) $\frac{7}{15} - \frac{1}{6}$

c) $\frac{5}{8} - \frac{1}{3}$

d) $\frac{11}{16} - \frac{7}{12}$

e) $\frac{5}{6} - \frac{8}{15}$

- Check your answers with those given at the end of the topic.
- If you are not sure how to add or subtract rational numbers, or if you had more than one error in each CHECK EXERCISE, check <u>Modern School Mathematics</u> pp. 336-340 or ask a student helper or ask your teacher.
- Otherwise, go on to section 2.

OBJECTIVE B2.1

(1) To write a fraction with numerator greater than denominator as a mixed numeral.

(2) To write a mixed numeral as a fraction.

Example

(1) Write as a mixed numeral $\frac{35}{8}$

(2) Write as a fraction $7\frac{2}{3}$

SOLUTION

$$\frac{35}{8} = \frac{32}{8} + \frac{3}{8} = 4\frac{3}{8}$$

$$7\frac{2}{3} = \frac{21}{3} + \frac{2}{3} = \frac{23}{3}$$

Criterion: Correct mixed numeral and fraction.

OBJECTIVE I2.2

a) To use fractions to justify that a given fraction has a particular mixed numeral.

b) To use fractions to justify that a given mixed numeral has a particular fraction.

Example

a) Use fractions to justify that the mixed numeral for $\frac{22}{5}$ *is* $4\frac{2}{5}$.

b) Use fractions to justify that the fraction for $5\frac{2}{3}$ *is* $\frac{17}{3}$.

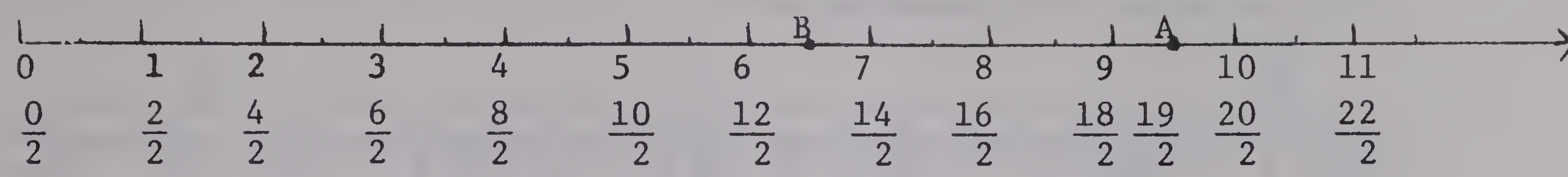
Criterion: Correct justifications.

SOLUTION

a) $\frac{22}{5} = \frac{20}{5} + \frac{2}{5} = \frac{20 \div 5}{5 \div 5} + \frac{2}{5} = \frac{4}{1} + \frac{2}{5} = 4 + \frac{2}{5} = 4\frac{2}{5}$

b) $5\frac{2}{3} = 5 + \frac{2}{3} = \frac{5}{1} + \frac{2}{3} = \frac{5 \times 3}{1 \times 3} + \frac{2}{3} = \frac{15}{3} + \frac{2}{3} = \frac{17}{3}$

Section 2. Mixed numerals and fractions



1. Write two ways of naming the rational number associated with point B on the number line.

The rational number associated with point A on the number line above may be named in two ways; $\frac{19}{2}$ or $9\frac{1}{2}$. The second of the names is the one often used in everyday situations; e.g. $9\frac{1}{2}$ inches, $9\frac{1}{2}$ years, etc.

On the number line subdivided into halves, $9\frac{1}{2}$ means 9 whole divisions plus one of the halves subdivisions. This is the same as 19 subdivisions, each one half.

$9\frac{1}{2}$ is called a mixed numeral because it contains both a whole number numeral and a fraction.

$9\frac{1}{2}$ fraction	$9\frac{1}{2}$	$9\frac{1}{2} = 9 + \frac{1}{2}$
whole number numeral	mixed numeral	meaning

When we get to applications of rational numbers to answer questions about everyday situations we will find that mixed numerals are often used. We will also need to be able to convert between mixed numerals and fractions so let's see how this is done.

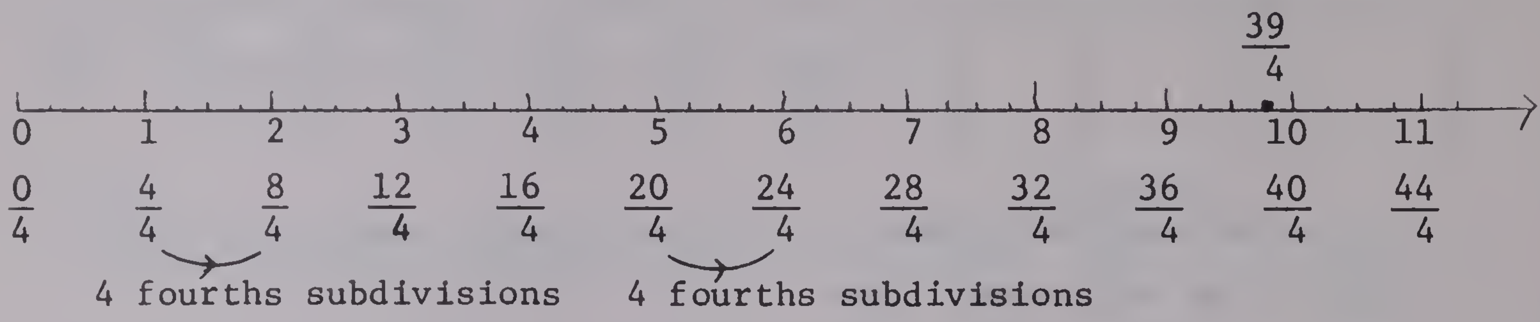
Fractions to mixed numerals

Of course, it is only when the numerator of the fraction is greater than the denominator that this can be done.

2. Write the mixed numeral for $\frac{17}{4}$.

Answers: 1: $\frac{13}{2}$ or $6\frac{1}{2}$ 2: 16 fourths are 4. Thus $\frac{17}{4} = 4\frac{1}{4}$.

Consider the fraction $\frac{39}{4}$. In terms of a number line subdivided into fourths, the point corresponding to $\frac{39}{4}$ is 39 fourths subdivisions from the point corresponding to 0.



It takes 4 fourths subdivisions to make each whole division. How many whole subdivisions are included in 39 fourths subdivisions? We find out by dividing 39 by 4.

$$\begin{array}{r} 9 \\ 4\overline{)39} \\ \underline{36} \\ 3 \end{array}$$

39 fourths subdivisions is 9 whole divisions and 3 fourths subdivisions.

i.e. $\frac{39}{4} = 9\frac{3}{4}$

This can also be justified, using fractions, as follows:

$$\frac{39}{4} = \frac{36}{4} + \frac{3}{4} = \frac{36 \div 4}{4 \div 4} + \frac{3}{4} = \frac{9}{1} + \frac{3}{4} = 9 + \frac{3}{4} = 9\frac{3}{4}$$

Let's try $\frac{104}{16}$ and obtain the mixed numeral for it.

First divide, to find the number of wholes.

$$\begin{array}{r} 6 \\ 16\overline{)104} \\ \underline{96} \\ 8 \end{array} \qquad 6\frac{8}{16}$$

But $\frac{8}{16}$ can be reduced to the basic fraction $\frac{1}{2}$

Thus $\frac{104}{16} = 6\frac{8}{16} = 6\frac{1}{2}$

Justification using fractions

$$\begin{aligned} \frac{104}{16} &= \frac{96}{16} + \frac{8}{16} \\ &= \frac{96 \div 16}{16 \div 16} + \frac{8 \div 8}{16 \div 8} \\ &= \frac{6}{1} + \frac{1}{2} \\ &= 6 + \frac{1}{2} \\ &= 6\frac{1}{2} \end{aligned}$$

3. Write $\frac{100}{8}$ as a mixed numeral.

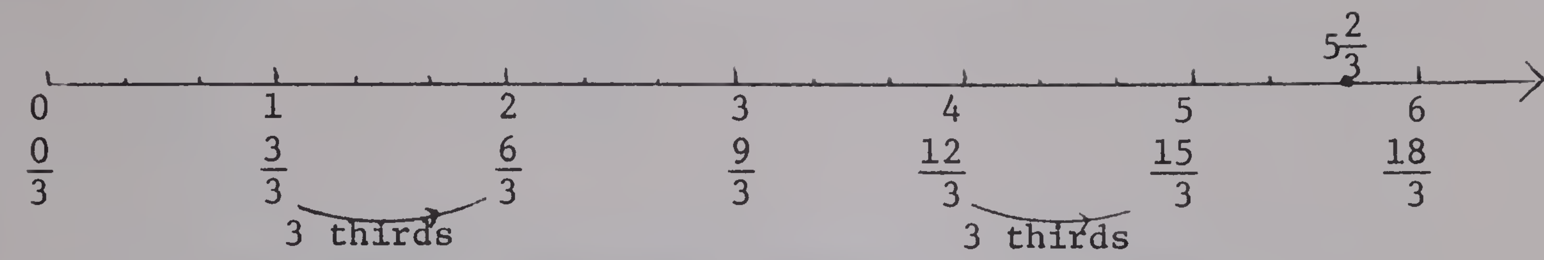
<u>Mixed numerals to fractions</u>

This can be done for every mixed numeral.

Answers: 3: $\begin{array}{r} 12 \\ 8\overline{)100} \end{array}$ remainder 4 $\qquad \frac{100}{8} = 12\frac{4}{8} = 12\frac{1}{2}$

4. Write the fraction for $4\frac{2}{3}$.

Consider the mixed numeral $5\frac{2}{3}$.



Each whole division has 3 thirds subdivisions.

5 whole divisions have 5 x 3 = 15 thirds subdivisions.

$5\frac{2}{3}$ has 15 thirds subdivisions plus 2 more thirds subdivisions;

i.e. 17 thirds subdivisions.

Thus $5\frac{2}{3} = \frac{17}{3}$

This can also be justified using fractions as follows:

$$5\frac{2}{3} = 5 + \frac{2}{3} = \frac{5}{1} + \frac{2}{3} = \frac{5 \times 3}{3 \times 3} + \frac{2}{3} = \frac{15}{3} + \frac{2}{3} = \frac{17}{3}$$

Let's try $4\frac{7}{12}$ and obtain the fraction for it.

First multiply to find the number of twelfths in 4

4 is 4 x 12 = 48 twelfths

$4\frac{7}{12}$ is (48 + 7) twelfths

$4\frac{7}{12} = \frac{55}{12}$

Justification using fractions

$$\begin{aligned} 4\frac{7}{12} &= 4 + \frac{7}{12} \\ &= \frac{4}{1} + \frac{7}{12} \\ &= \frac{4 \times 12}{1 \times 12} + \frac{7}{12} \\ &= \frac{48}{12} + \frac{7}{12} \\ &= \frac{55}{12} \end{aligned}$$

5. Write the mixed numeral for $9\frac{8}{9}$.

For a number to be correctly named by a mixed numeral, the fraction part must have numerator less than denominator.

$3\frac{5}{4}$ is not written correctly. It should be $4\frac{1}{4}$.

Answers:4: 4 is 12 thirds. Thus $4\frac{2}{3}$ is $\frac{14}{3}$.

5: 9 is 9 x 9 = 81 ninths. $9\frac{8}{9} = \frac{89}{9}$

Now read OBJECTIVES B2.1 and I2.2 and their examples. These tell you what you are expected to be able to do for this section. When ready, turn to and do CHECK EXERCISES 2.1 and 2.2.

CHECK EXERCISE 2

B2.1 i) Write the following fractions as mixed numerals:

a) $\frac{37}{5}$ d) $\frac{13}{7}$

b) $\frac{9}{4}$ e) $\frac{16}{9}$

c) $\frac{11}{3}$

ii) Write the following mixed numerals as fractions:

a) $1\frac{3}{4}$ d) $4\frac{4}{9}$

b) $2\frac{5}{7}$ e) $5\frac{3}{8}$

c) $1\frac{8}{11}$

I2.2 i) Use fractions to justify that the mixed numeral for:

a) $\frac{15}{4}$ is $3\frac{3}{4}$

b) $\frac{22}{7}$ is $3\frac{1}{7}$

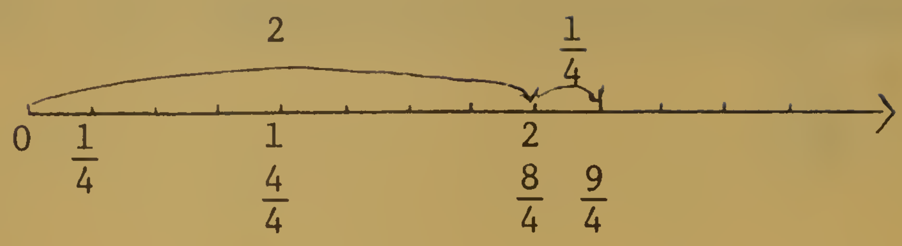
ii) Use fractions to justify that the fraction for:

a) $3\frac{3}{5}$ is $\frac{18}{5}$

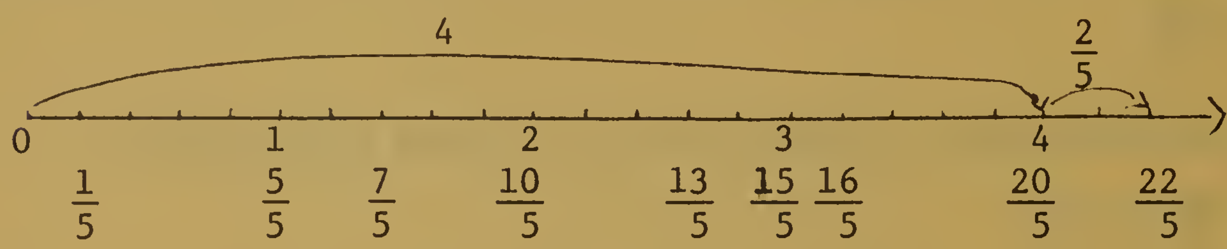
b) $2\frac{1}{4}$ is $\frac{9}{4}$

- Check your answers with those given at the end of the topic.
  You were successful on CHECK EXERCISE B2.1 if you had at least 4 of the parts correct in both 2.1 (i) and 2.1 (ii).
  You were successful on CHECK EXERCISE I2.2 if you had 3 of the 4 parts correct and all steps were shown.
- If you are not sure how to do either of the above CHECK EXERCISES or if you were unsuccessful on CHECK EXERCISE B2.1 read section 2 and do activity exercise 2.1.
  If you were unsuccessful on CHECK EXERCISE I2.2 read section 2 and do activity exercise 2.2.
- Otherwise, go on to section 3.

Activity Exercise 2.1



From the number line, we can see that $\frac{9}{4}$ means 2 whole units on the number line plus one-quarter of the next unit. This could be written as $2\frac{1}{4}$, which means $2 + \frac{1}{4}$. $2\frac{1}{4}$ is said to be a <u>mixed numeral</u> as it contains both a whole number and a fraction.



On the number line above we see that $\frac{22}{5}$ has four whole units divided into fifths plus $\frac{2}{5}$ of the next unit. ie. $\frac{22}{5} = 4\frac{2}{5}$

1. Using the above number line, write mixed numerals for:

   a) $\frac{7}{5}$ b) $\frac{13}{5}$ c) $\frac{16}{5}$

   Of course we do not want to always draw a number line to determine the mixed numeral. We can use the following method which involves division.

   We can find the mixed numeral for $\frac{22}{5}$ by dividing 22 by 5.

$$\begin{array}{r} 4 \\ 5\overline{)\,22} \\ \underline{20} \\ 2 \end{array} \quad \rightarrow 4\frac{2}{5}$$

   Four whole units and $\frac{2}{5}$ of the next unit.

   Similarly $\frac{26}{5} = 5\frac{1}{5}$ because

$$\begin{array}{r} 5 \\ 5\overline{)\,26} \\ \underline{25} \\ 1 \end{array} \quad \text{i.e. } 5\frac{1}{5}$$

2. Using the division method write mixed numerals for:

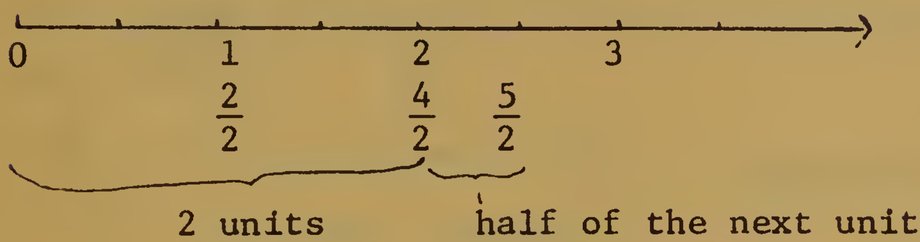
a) $\frac{15}{4}$ d) $\frac{31}{6}$

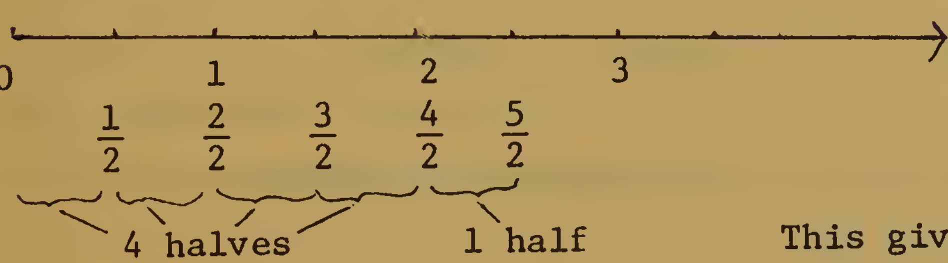
b) $\frac{17}{3}$ e) $\frac{24}{7}$

c) $\frac{21}{8}$

Given the mixed numeral we also want to be able to express it as a fraction.

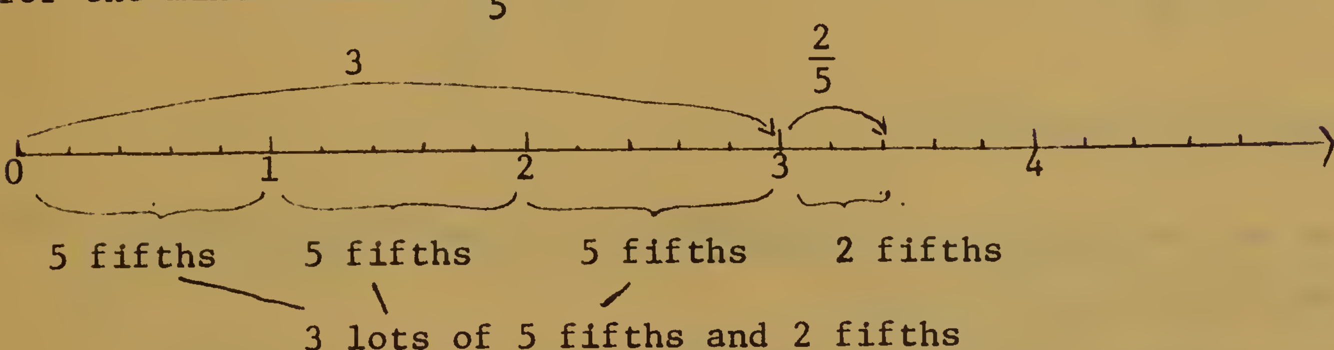
$2\frac{1}{2}$ means $2 + \frac{1}{2}$ or 2 units plus $\frac{1}{2}$ of the next unit.





This gives 5 halves or $\frac{5}{2}$.

Study the following example in which we write a fraction for the mixed numeral $3\frac{2}{5}$.



i.e. (3 x 5) fifths + 2 fifths

i.e. $\frac{15}{5} + \frac{2}{5}$

i.e. $\frac{17}{5}$

3. Write a fraction for $2\frac{3}{4}$.

Again we want to be able to write fractions equivalent to mixed numerals without having to draw a number line. Looking at the mixed numeral $3\frac{2}{5}$ we see that it is made up of 3 and $\frac{2}{5}$. To find the number of fifths in 3 we multiply 3 by 5. This gives us 15 fifths in the 3 plus 2 more fifths in the $\frac{2}{5}$. Thus we have 15 + 2 fifths, i.e. 17 fifths or $\frac{17}{5}$.

To do this quivkly we can think $3\frac{2}{5}$ = [(5 x 3) + 2] fifths

= [15 + 2] fifths

= $\frac{17}{5}$

$4\frac{2}{3}$ = [(3 x 4) + 2] thirds

= 12 + 2 thirds

= $\frac{14}{3}$

4. Write fractions for the following mixed numerals:

a) $5\frac{2}{5}$

b) $3\frac{2}{3}$

c) $1\frac{4}{7}$

d) $5\frac{1}{4}$

e) $2\frac{7}{9}$

Activity Exercise 2.2

We can use <u>fractions</u> to <u>justify</u> that a given fraction has a particular mixed numeral.

For example:

$$\begin{aligned}\frac{15}{4} &= \frac{12}{4} + \frac{3}{4}\\ &= \frac{12 \div 4}{4 \div 4} + \frac{3}{4}\\ &= \frac{3}{1} + \frac{3}{4}\\ &= 3 + \frac{3}{4}\\ &= 3\frac{3}{4}\end{aligned}$$

all these steps are necessary for the justification

1. Write out and complete the next example.

$$\frac{23}{5} = \frac{\square}{5} + \frac{\square}{5}$$
$$= \frac{20 \div \square}{5 \div \square} + \frac{3}{5}$$
$$= \frac{\square}{\square} + \frac{3}{5}$$
$$= \square + \square$$
$$= \square$$

(note: we want to form two fractions such that one may be simplified to a whole number and the other has numerator less than denominator).

2. Justify that each of the following fractions is the given mixed numeral:

a) $\frac{37}{6}$ is the mixed numeral $6\frac{1}{6}$.

b) $\frac{17}{3}$ is the mixed numeral $5\frac{2}{3}$.

We can also use fractions to justify that a given mixed numeral has a particular fraction.

For example:

$$4\frac{2}{3} = 4 + \frac{2}{3}$$
$$= \frac{4}{1} + \frac{2}{3}$$
$$= \frac{4 \times 3}{1 \times 3} + \frac{2}{3}$$
$$= \frac{12}{3} + \frac{2}{3}$$
$$= \frac{14}{3}$$

all these steps are needed for the justification

3. Write out and complete the following example.

$$3\frac{5}{7} = \square + \square$$
$$= \frac{\square}{\square} + \frac{5}{7}$$
$$= \frac{3 \times \square}{1 \times \square} + \frac{5}{7}$$
$$= \frac{\square}{\square} + \frac{5}{7}$$
$$= \frac{\square}{\square}$$

(note: we want to write the whole number as a fraction with the same denominator as the fraction in the mixed numeral).

OBJECTIVE B3.1

To find the sum of two or more rational numbers named by fractions or mixed numerals.

Example

Find the sums and write each as a mixed numeral with fraction part as a basic fraction.

A. $\begin{array}{r} 4\frac{1}{8} \\ +\ 2\frac{1}{4} \\ \hline \end{array}$

B. $5\frac{1}{2} + 2\frac{2}{3} + \frac{3}{4}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A.

$$\begin{array}{rcl} 4\frac{1}{8} & = & 4 + \frac{1}{8} \\ 2\frac{1}{4} & = & 2 + \frac{2}{8} \\ \hline & & 6\frac{3}{8} \end{array}$$

B.

$$\begin{aligned} 5\frac{1}{2} + 2\frac{2}{3} + \frac{3}{4} &= 5\frac{6}{12} + 2\frac{8}{12} + \frac{9}{12} \\ &= 7 + \frac{23}{12} \\ &= 7 + 1\frac{11}{12} \\ &= 8\frac{11}{12} \end{aligned}$$

OBJECTIVE I3.1

To find the sum of two or more rational numbers named by fractions or mixed numerals.

Example

Find the sums and write each as a mixed numeral with fraction part as a basic fraction.

A. $\begin{array}{r} 1\frac{1}{4} \\ \frac{2}{5} \\ +\ 1\frac{2}{3} \\ \hline \end{array}$

B. $\frac{4}{5} + 3 + 4\frac{1}{3} + \frac{7}{10}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A.

$$\begin{aligned} 1\tfrac{1}{4} &= 1 + \tfrac{15}{60} \\ \tfrac{2}{5} &= \tfrac{24}{60} \\ 1\tfrac{2}{3} &= 1 + \tfrac{40}{60} \\ &= 1 + \tfrac{79}{60} = 1 + 1 + \tfrac{19}{60} = 2\tfrac{19}{60} \end{aligned}$$

B.

$$\begin{aligned} \tfrac{4}{5} + 3 + 4\tfrac{1}{3} + \tfrac{7}{10} &= \tfrac{24}{30} + 3 + 4\tfrac{10}{30} + \tfrac{21}{30} \\ &= 7 + \tfrac{55}{30} \\ &= 7 + 1\tfrac{25}{30} \\ &= 8\tfrac{25}{30} \\ &= 8\tfrac{5}{6} \end{aligned}$$

OBJECTIVE B3.2

To find the difference of two rational numbers named by fractions or mixed numerals.

Example

Find the differences and write the fraction parts as basic fractions.

A. $3\frac{2}{3} - 1\frac{2}{5}$

B. $\begin{array}{r} 4\frac{1}{4} \\ -\ 3\frac{5}{6} \\ \hline \end{array}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A.

$$\begin{aligned} 3\tfrac{2}{3} - 1\tfrac{2}{5} &= 3\tfrac{10}{15} - 1\tfrac{6}{15} \\ &= 2\ \tfrac{4}{15} \end{aligned}$$

B.

$$\begin{array}{rcrcr} 4\frac{1}{4} & = & 4 + \frac{3}{12} & = & 3 + \frac{15}{12} \\ -\ 3\frac{5}{6} & = & 3 + \frac{10}{12} & = & 3 + \frac{10}{12} \\ \hline & & & & \frac{5}{12} \end{array}$$

OBJECTIVE I3.2

To find the difference of two rational numbers named by fractions or mixed numerals.

Example

Find the differences and write the fraction parts as basic fractions:

A. $3\frac{5}{8} - 2\frac{4}{5}$

B. $\begin{array}{r} 6\frac{5}{9} \\ -\ \frac{3}{4} \\ \hline \end{array}$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

A.

$$
\begin{aligned}
3\frac{5}{8} - 2\frac{4}{5} &= 3\frac{25}{40} - 2\frac{32}{40} \\
&= 1 + \frac{25}{40} - \frac{32}{40} \\
&= \frac{65}{40} - \frac{32}{40} \\
&= \frac{33}{40}
\end{aligned}
$$

B.

$$
\begin{array}{rcrcr}
6\frac{5}{9} & = & 6 + \frac{20}{36} & = & 5 + \frac{56}{36} \\
- \;\; \frac{3}{4} & = & \frac{27}{36} & = & \frac{27}{36} \\
\hline
 & & & & 5\frac{29}{36}
\end{array}
$$

Section 3. Addition and Subtraction involving rational numbers named by mixed numerals

Example 1 Read through this example and the comments

$3\frac{1}{4} + 2\frac{2}{3}$

$= 3 + \frac{1}{4} + 2 + \frac{2}{3}$

Mixed numerals represent the sum of a number named by a whole number numeral and a number named by a fraction.

$= 3 + 2 + \frac{1}{4} + \frac{2}{3}$

$= 5 + \frac{3}{12} + \frac{8}{12}$

Add the whole numbers and add the numbers named by the fractions. The L.C.D. is 12.

$= 5 + \frac{11}{12}$

$= 5\frac{11}{12}$

Write the result as a mixed numeral.

You can probably do an example like this without writing down steps 2, 3 or 5

$3\frac{1}{4} + 2\frac{2}{3}$

$= 5 + \frac{3}{12} + \frac{8}{12}$

$= 5\frac{11}{12}$

Example 2 Read through this example and the comments

$3\frac{3}{4} + 2\frac{2}{3}$

$= 5 + \frac{9}{12} + \frac{8}{12}$

Add the whole numbers and add the numbers named by the fractions. The L.C.D. is 12.

$= 5 + \frac{17}{12}$

$\frac{17}{12}$ is $1\frac{5}{12}$ as a mixed numeral, i.e. $1 + \frac{5}{12}$

$= 5 + 1 + \frac{5}{12}$

$= 6\frac{5}{12}$

Add the whole numbers.

1. Find the sum: $5\frac{3}{4} + \frac{5}{6}$

Answer: 1: $5\frac{3}{4} + \frac{5}{6}$

$= 5 + \frac{9}{12} + \frac{10}{12}$

$= 5 + \frac{19}{12}$

$= 5 + 1 + \frac{7}{12}$

$= 6\frac{7}{12}$

Example 3   Read through this example and the comment

$$4\tfrac{2}{3} - 1\tfrac{1}{2}$$

$$= 4 + \frac{2}{3} - (1 + \frac{1}{2})$$

$$= 4 - 1 + \frac{2}{3} - \frac{1}{2}$$

$$= 3 + \frac{4}{6} - \frac{3}{6}$$

Subtract the whole numbers and subtract the numbers named by the fractions. The L.C.D. is 6.

$$= 3 + \frac{1}{6}$$

$$= 3\tfrac{1}{6}$$

You can probably do an example like this without writing down steps 2, 3 or 5.

Example 4   Read through this example and the comments

Sometimes in subtraction it is necessary to "borrow". This happens when the number to be subtracted is greater than the number it is to be subtracted from.

$$4\tfrac{1}{3} - 1\tfrac{3}{4}$$

$$= 3 + \frac{4}{12} - \frac{9}{12}$$

Subtract the whole numbers. When attempting to subtract the numbers named by fractions we see that $\frac{9}{12}$ is greater than $\frac{4}{12}$.

$$= 2 + 1\tfrac{4}{12} - \frac{9}{12}$$

"Borrow 1" from the whole number

$$= 2 + \frac{16}{12} - \frac{9}{12}$$

$1\frac{4}{12} = \frac{16}{12}$. Now we can subtract.

$$= 2 + \frac{7}{12}$$

$$= 2\tfrac{7}{12}$$

2. Subtract: $5\frac{3}{8} - 4\frac{5}{6}$

Answer: 2: $5\frac{3}{8} - 4\frac{5}{6}$

$$= 1 + \frac{9}{24} - \frac{20}{24}$$

$$= 1\tfrac{9}{24} - \frac{20}{24}$$

$$= \frac{33}{24} - \frac{20}{24}$$

$$= \frac{13}{24}$$

(addition may also be done with the numbers arranged vertically).

<u>Examples 5 and 6</u> Read through these examples.

$$\begin{array}{rcl} 3\frac{1}{4} & = & 3 + \frac{3}{12} \\ 2\frac{1}{3} & = & 2 + \frac{4}{12} \\ + \ 4\frac{1}{6} & = & 4 + \frac{2}{12} \\ \hline & & 9 + \frac{9}{12} \\ & = & 9\frac{3}{4} \end{array}$$

L.C.D. is 12

Basic fraction for $\frac{9}{12}$ is $\frac{3}{4}$

$$\begin{array}{rcl} 3\frac{3}{4} & = & 3 + \frac{18}{24} \\ \frac{5}{6} & = & \frac{20}{24} \\ + \ 1\frac{7}{8} & = & 1 + \frac{21}{24} \\ \hline & & 4 + \frac{59}{24} \\ & = & 4 + 2 + \frac{11}{24} \\ & = & 6\frac{11}{24} \end{array}$$

$\frac{59}{24} = 2\frac{11}{24}$

3. Add:

$$\begin{array}{r} \frac{9}{10} \\ 2\ \frac{1}{4} \\ + \ 1\ \frac{2}{5} \\ \hline \end{array}$$

<u>Example 7</u> Read through this example

$$\begin{array}{rcccccc} 12\frac{1}{3} & = & 12 + \frac{8}{24} & = & 11 + 1\frac{8}{24} & = & 11 + \frac{32}{24} \\ - \ 3\frac{5}{8} & = & 3 + \frac{15}{24} & = & 3 + \frac{15}{24} & = & 3 + \frac{15}{24} \\ \hline & & & & & & 8 + \frac{17}{24} \\ & & & & & = & 8\frac{17}{24} \end{array}$$

$\frac{15}{24}$ is greater than $\frac{8}{24}$.

We need to "borrow 1" from the 12.

4. Subtract

$$\begin{array}{r} 4\frac{1}{6} \\ - \ 3\frac{4}{5} \\ \hline \end{array}$$

Now read Objectives I3.1 and I3.2 and their examples. These tell you what you are expected to be able to do for this section. When ready, turn to and do CHECK EXERCISES 3.1 and 3.2.

Answers: 3:

$$\begin{array}{rcl} \frac{9}{10} & = & \frac{18}{20} \\ 2\frac{1}{4} & & 2 + \frac{5}{20} \\ + \ 1\frac{2}{5} & & 1 + \frac{8}{20} \\ \hline & & 3 + \frac{31}{20} \\ & = & 3 + 1 + \frac{11}{20} \\ & = & 4\frac{11}{20} \end{array}$$

4:

$$\begin{array}{rcccccc} 4\frac{1}{6} & = & 4 + \frac{5}{30} & = & 3 + 1\frac{5}{30} & = & 3 + \frac{35}{30} \\ - \ 3\frac{4}{5} & = & 3 + \frac{24}{30} & = & 3 + \frac{24}{30} & = & 3 + \frac{24}{30} \\ \hline & & & & & & \frac{11}{30} \end{array}$$

CHECK EXERCISE 3

I3.1 Find the sums and write each as a mixed numeral with fractional part as a basic fraction:

a) $3\frac{9}{10} + 2\frac{3}{8} + 1\frac{3}{4}$

b) $4\frac{2}{3} + 3\frac{1}{5} + 5\ \frac{7}{12}$

c) $4\frac{1}{3} + 2\frac{3}{10} + 2\frac{1}{2}$

d) $\begin{array}{r} 1\frac{3}{4} \\ 2\frac{5}{12} \\ +\ \frac{7}{2} \\ \hline \end{array}$

e) $\begin{array}{r} 9\frac{1}{5} \\ \frac{1}{3} \\ +\ 8 \\ \hline \end{array}$

I3.2 Find the differences and write the fractional parts as basic fractions:

a) $8\frac{5}{6} - 4\frac{3}{10}$

b) $7\frac{1}{6} - 4\frac{3}{4}$

c) $17\frac{5}{8} - 5\frac{1}{4}$

d) $\begin{array}{r} 91\frac{4}{7} \\ -\ 39\frac{7}{8} \\ \hline \end{array}$

e) $\begin{array}{r} 45\frac{5}{9} \\ -\ 37\frac{5}{6} \\ \hline \end{array}$

- Check your answers with those given at the end of the topic. You are successful in each CHECK EXERCISE if you are using the correct method and have no more than one part incorrect in each question.
- If you are not sure how to add mixed numerals or if you were unsuccessful with CHECK EXERCISE 3.1, read section 3 and do Activity exercises 3.1.
- If you are not sure how to subtract mixed numerals or if you were unsuccessful with CHECK EXERCISE 3.2 read section 3 and do Activity exercises 3.2.
- Otherwise go on to Section 4.

Activity Exercise 3.1

To add $4\frac{1}{5} + 3\frac{2}{5}$ we first write each mixed numeral as the sum

$4 + \frac{1}{5} + 3 + \frac{2}{5}$ of a whole number and a number named by a fraction

$4 + 3 + \frac{1}{5} + \frac{2}{5}$ We add the whole numbers and then the numbers named by the fractions.

$7 + \frac{3}{5}$ We write the result

$7\frac{3}{5}$

1. Add $5\frac{3}{7} + 3\frac{2}{7}$

Carefully read through the following examples and comments:

a)* $2\frac{3}{8} + 5\frac{3}{4} + 3\frac{5}{6}$

$= 2 + \frac{3}{8} + 5 + \frac{3}{4} + 3 + \frac{5}{6}$

$= 2 + 5 + 3 + \frac{3}{8} + \frac{3}{4} + \frac{5}{6}$ — in order to add the numbers named by fractions we need to express them with their L.C.D. of 24.

*$= 10 + \frac{9}{24} + \frac{18}{24} + \frac{20}{24}$

*$= 10 + \frac{47}{24}$

$= 10 + 1\frac{23}{24}$ — the mixed numeral for $\frac{47}{24}$ is $1\frac{23}{24}$

$= 10 + 1 + \frac{23}{24}$

$= 11 + \frac{23}{24}$

*$= 11\frac{23}{24}$ — expressed as a mixed numeral

* When doing such an example you would probably only need to write down the lines marked with an *.

b) * $5\frac{5}{6} + 6\frac{1}{2}$

$= 11 + \frac{5}{6} + \frac{1}{2}$

*$= 11 + \frac{5}{6} + \frac{3}{6}$ — L.C.D. is 6

*$= 11 + \frac{8}{6}$ — $\frac{8}{6}$ is the mixed numeral $1\frac{2}{3}$

$= 11 + 1 + \frac{2}{6}$

*$= 12\frac{2}{6}$

*$= 12\frac{1}{3}$ — $\frac{2}{6}$ is the basic fraction $\frac{1}{3}$

Note: Always express the mixed numeral with a BASIC FRACTION.

2. Add the following:

a) $3\frac{9}{10} + 4\frac{4}{5}$

b) $11\frac{7}{10} + 16\frac{3}{4} + 5\frac{1}{2}$

c) $7\frac{1}{4} + 3\frac{7}{8} + 4\frac{5}{6}$

d) $\begin{array}{r} 2\frac{1}{2} \\ \frac{3}{5} \\ + 1\frac{1}{2} \\ \hline \end{array}$

e) $\begin{array}{r} 2\frac{3}{5} \\ 3\frac{4}{7} \\ + 5\frac{3}{10} \\ \hline \end{array}$

Activity Exercise 3.2

$$4\tfrac{5}{7} - 2\tfrac{3}{7}$$

$= 4 + \frac{5}{7} - (2 + \frac{3}{7})$ subtract whole numbers and numbers named by fractions.

$= 4 - 2 + \frac{5}{7} - \frac{3}{7}$

$= 2 + \frac{2}{7}$

$= 2\frac{2}{7}$ express as a mixed numeral

1. Find the difference:

$$8\tfrac{4}{5} - 5\tfrac{1}{5}$$

In order to subtract $7\frac{3}{4} - 5\frac{1}{8}$ we need to express the fractions with their L.C.D.

Thus $7\frac{3}{4} - 5\frac{1}{8}$ (You can do several steps at once.)

$= 7 - 5 + \frac{6}{8} - \frac{1}{8}$ fractions expressed with L.C.D. of 8.

$= 2 + \frac{5}{8}$

$= 2\frac{5}{8}$ expressed as a mixed numeral.

2. Find the difference:

$$5\tfrac{2}{3} - 2\tfrac{2}{5}$$

Here is another example:

$$\begin{array}{r} 27\frac{1}{3} \\ -18\frac{5}{6} \\ \hline \end{array}$$

$$\begin{array}{r} 27\frac{1}{3} \\ -\ 18\frac{5}{6} \\ \hline \end{array} \quad = \quad \begin{array}{r} 27 + \frac{2}{6} \\ -\ 18 + \frac{5}{6} \\ \hline \end{array}$$

We have expressed the fractions in the mixed numerals with common denominators. But we find we cannot subtract $\frac{5}{6}$ from $\frac{2}{6}$. It is too big. Thus we must "borrow" one from the whole number.

$$\begin{aligned} 27 + \frac{2}{6} &= 26 + 1 + \frac{2}{6} \\ &= 26 + 1\frac{2}{6} \\ &= 26 + \frac{8}{6} \end{aligned}$$

Now we can subtract $\frac{5}{6}$ from $\frac{8}{6}$. We have

$$\begin{array}{rcrcr} 27\frac{1}{3} & = & 27 + \frac{2}{6} & = & 26 + \frac{8}{6} \\ -\ 18\frac{5}{6} & = & 18 + \frac{5}{6} & = & 18 + \frac{5}{6} \\ \hline & & & & 8 + \frac{3}{6} \\ & & & = & 8\frac{1}{2} \end{array}$$

a mixed numeral with a basic fraction.

3. Find the differences of the following and write the fractions as basic fractions.

a) $6\frac{1}{2} - 5\frac{1}{4}$

b) $4\frac{1}{4} - 3\frac{3}{16}$

c) $5\frac{1}{4} - 2\frac{3}{8}$

d) $\begin{array}{r} 4\frac{3}{16} \\ -\ 3\frac{1}{4} \\ \hline \end{array}$

e) $\begin{array}{r} 8\frac{3}{10} \\ -\ 5\frac{5}{7} \\ \hline \end{array}$

- Check your answers with those given at the end of the topic.
- Read objectives I3.1 and I3.2.
- If you feel you are ready, do CHECK EXERCISE 3A.

CHECK EXERCISE 3A

I3.1 Find the sums and write each as a mixed numeral with fraction as a basic fraction:

a) $3\frac{4}{5} + 7\frac{3}{10} + 2\frac{1}{2}$

b) $8\frac{2}{3} + \frac{7}{8} + 6\frac{1}{4}$

c) $3\frac{7}{9} + 5\frac{5}{6} + 2\frac{3}{4}$

d) $\begin{array}{r} 4\frac{3}{7} \\ 2\frac{5}{6} \\ +\ 5\frac{1}{4} \\ \hline \end{array}$

e) $\begin{array}{r} 1\frac{7}{15} \\ \frac{4}{5} \\ +\ 8\frac{5}{12} \\ \hline \end{array}$

I3.2 Find the differences and express the fractions as basic fractions:

a) $3\frac{1}{2} - 2\frac{1}{4}$

b) $4\frac{1}{3} - 2\frac{5}{12}$

c) $16\frac{1}{4} - 15\frac{3}{5}$

d) $\begin{array}{r} 2\frac{1}{7} \\ -\ 1\frac{9}{14} \\ \hline \end{array}$

e) $\begin{array}{r} 13\frac{5}{6} \\ -\ 12\frac{11}{12} \\ \hline \end{array}$

- Check your answers with those given at the end of the topic. You are successful if you use the correct method and you have no more than one part incorrect in each CHECK EXERCISE. 166
- If you are not sure how to add or subtract rational numbers named by mixed numerals, or if you were unsuccessful on either CHECK EXERCISE, check reference book Modern School Mathematics, pp 346, 347 or ask a student helper

  or ask your teacher.
- Otherwise, go on to section 4.

Summary

Section 1

Addition and subtraction of rational numbers named by fractions is usually done by obtaining equivalent fractions with least common denominators and then adding or subtracting the numerators. Answers are given as basic fractions.

Section 2

A mixed numeral contains a whole number numeral and a fraction.

Each mixed numeral has a corresponding fraction

$$4\frac{3}{8} = (4 \times 8) + 3 \text{ eighths}$$
$$= 35 \text{ eighths}$$
$$= \frac{35}{8}$$

AND

Each fraction with numerator greater than denominator has a corresponding mixed numeral.

$$\frac{35}{8} = 8\overline{)35} = 4\frac{3}{8}$$

Section 3

In addition (or subtraction) of rational numbers named by mixed numerals, the whole numbers are added (or subtracted) and the numbers named by the fractions are added (or subtracted).

Section 4

A solution for a condition for equality is a number, which when put in place of the variable, makes the condition a true statement.

Conditions for equality involving addition (or subtraction) of rational numbers are solved by finding (related conditions with the variable alone on one side.

Section 5

Addition and subtraction of rational numbers are used to answer questions about every day situations. Most examples are done by

(1) finding the relationship between what is asked and what is given.

(2) obtaining the relationship in mathematical form and doing the indicated mathematical work to get a mathematical answer.

(3) Using the mathematical answer to answer the original question.

Section 6

$$\frac{a}{b} \text{ of } \frac{c}{d} = \frac{a \times c}{b \times d}$$

Section 7

Multiplication of rational numbers is done as follows:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

When mixed numerals or whole numbers are involved, their corresponding fractions are used. Answers are given as basic fractions. Reduction to basic fractions is done by dividing numerator and denominator by their common factors (cancelling).

Section 8

The reciprocal of a number is another number whose product with the given number is 1.

The reciprocal of a non-zero rational number is obtained by interchanging the numerator and denominator of the fraction for the rational number.

Zero has no reciprocal.

Section 9

To divide by a non-zero rational number, you multiply by its reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

It is not possible to divide by 0.

Section 10

Conditions for equality involving multiplication of rational numbers can often be solved by finding related conditions involving division of rational numbers with the variable alone on one side.

Also, in conditions for equality in which the variable is multiplied by a number, the variable can be obtained alone on one side by multiplying both sides by the reciprocal of the number by which the variable is multiplied.

Section 11

To answer questions from everyday situations by mathematics we usually find the relationship between what is asked for and what is given. The mathematical form of this relationship is called the mathematical model for the situation. The mathematical model is often a condition for equality. The solution for this condition for equality is used to answer the original question. 169

Section 12

The rational numbers are closed for addition and multiplication. Addition and multiplication of rational numbers are commutative and associative.

0 and 1 are the identities for addition and multiplication respectively of rational numbers.

Multiplication of rational numbers is distributive over addition. Non-zero rational numbers have reciprocals. This is a property of rational numbers which whole numbers do not have.

Vocabulary

analyse - with respect to a question about an every day situation it means to find the relationship between what is asked for and what is given. (page) (section 5, p 2)

application - use of mathematics to answer a question about an everyday situation. (page) (section 5,p 1)

associative - the property of an operation which permits a change in the grouping of the numbers to not affect the result (page) (section 12,p 1)

binary - a binary operation is done with just two numbers at a time. (page) (section 12, p 4)

cancel - to divide numerator and denominator of a fraction by a common factor. (page) (section 7, p 3)

closed - the property of an operation which states that the number resulting from the operation is in the same set as the numbers the operation is done with. (page) (section 12, p 1)

commutative - the property of an operation which permits a change in the order of the numbers to not affect the result. (page) (section 12, p 1)

complex fraction - a fraction like $\frac{\frac{2}{3}}{\frac{3}{4}}$ which represents $\frac{2}{3} \div \frac{3}{4}$.

condition for equality - a mathematical statement involving a variable and the = symbol. (page) (section 4, p 1)

difference - the result of a subtraction. (page) (section 1, Objectives)

distributive - a property of multiplication and addition of numbers (page) (section 12, p 1)

L.C.D. - least common denominator. (page) (section 1, p 2)

mixed numeral - a numeral involving a whole number numeral and a fraction. (page) (section 2, p 1)

model - the mathematical form for a relationship. (page) (section 11, p 1)

operation - addition, subtraction, multiplication and division are examples of operations. (page) (Introduction)

product - the result of a multiplication. (page) (section 7, p 1)

property - a characteristic or quality. (page) (section 12, p 1)

quotient - the result of a division.

reciprocal - a number whose product with a given number is 1. (page) (section 8, p 1)

related condition - a condition with the same solution as the given condition. (page) (section 4, p 1)

replacement set - the set of numbers which a variable can represent; same as universe. (page) (section 4, p 1)

solution - a number which makes a condition a true statement. (page) (section 4, p 1)

solve - the process of finding the solution to a condition or problem. (page) (section 4, p 1)

sum - the result of an addition. (page) (section 4, p 1)

universe - the set of numbers which a variable can represent; same as replacement set. (page) (section 4, p 1)

variable - a letter which can represent any of the numbers in a given set. (page) (section 4, p 1)

CHECK EXERCISE 1

I1.1 a) $\frac{67}{50}$ b) $\frac{95}{48}$ c) $\frac{31}{15}$ d) $\frac{41}{20}$ e) $\frac{77}{30}$

I1.2 a) $\frac{1}{2}$ b) $\frac{3}{10}$ c) $\frac{5}{18}$ d) $\frac{27}{40}$ e) $\frac{11}{12}$

Activity exercise 1.1

1 a) $\frac{6}{7}$ b)i) $\frac{7}{9}$ ii) $\frac{7}{4}$ iii) $\frac{11}{10}$ iv) $\frac{4}{5}$ c)i) $\frac{5}{4}$ ii) $\frac{5}{6}$ iii) $\frac{44}{15}$
iv) $\frac{71}{63}$ v) $\frac{187}{70}$ d)i) $\frac{2}{3}$ ii) $\frac{9}{10}$ iii) $\frac{47}{40}$ e)i) 2 ii) $\frac{9}{8}$
iii) $\frac{53}{40}$ iv) $\frac{23}{15}$ v) $\frac{9}{4}$

Activity exercise 1.2

a)i) $\frac{1}{9}$ ii) $\frac{4}{5}$ iii) $\frac{2}{8}$ b)i) $\frac{25}{15}$ ii) $\frac{7}{24}$ iii) $\frac{11}{35}$
c)i) $\frac{3}{16}$ ii) $\frac{1}{4}$ iii) $\frac{1}{4}$ iv) $\frac{1}{12}$ v) $\frac{43}{35}$

CHECK EXERCISE 1A

I1.1 a) $\frac{8}{9}$ b) $\frac{71}{100}$ c) $\frac{7}{12}$ d) $\frac{5}{6}$ e) $\frac{23}{24}$

I1.2 a) $\frac{1}{3}$ b) $\frac{3}{10}$ c) $\frac{7}{24}$ d) $\frac{5}{48}$ e) $\frac{3}{10}$

CHECK EXERCISE 2

B2.1 i) a) $7\frac{2}{5}$ b) $2\frac{1}{4}$ c) $3\frac{2}{3}$ d) $1\frac{6}{7}$ e) $1\frac{7}{9}$

ii) a) $\frac{7}{4}$ b) $\frac{19}{7}$ c) $\frac{19}{11}$ d) $\frac{40}{9}$ e) $\frac{43}{8}$

I2.2 i) a)
$$\begin{aligned}\frac{15}{4} &= \frac{12}{4} + \frac{3}{4}\\ &= \frac{12 \div 4}{4 \div 4} + \frac{3}{4}\\ &= \frac{3}{1} + \frac{3}{4}\\ &= 3 + \frac{3}{4}\\ &= 3\frac{3}{4}\end{aligned}$$

b)
$$\begin{aligned}\frac{22}{7} &= \frac{21}{7} + \frac{1}{7}\\ &= \frac{21 \div 7}{7 \div 7} + \frac{1}{7}\\ &= \frac{3}{1} + \frac{1}{7}\\ &= 3 + \frac{1}{7}\\ &= 3\frac{1}{7}\end{aligned}$$

I2.2ii) a) $3\frac{3}{5} = 3 + \frac{3}{5}$
$= \frac{3}{1} + \frac{3}{5}$
$= \frac{3 \times 5}{1 \times 5} + \frac{3}{5}$
$= \frac{15}{5} + \frac{3}{5}$
$= \frac{18}{5}$

b) $2\frac{1}{4} = 2 + \frac{1}{4}$
$= \frac{2}{1} + \frac{1}{4}$
$= \frac{2 \times 4}{1 \times 4} + \frac{1}{4}$
$= \frac{8}{4} + \frac{1}{4}$
$= \frac{9}{4}$

Activity Exercise 2.1

1.a) $1\frac{2}{5}$ b) $2\frac{3}{5}$ c) $3\frac{1}{5}$ 2a) $3\frac{3}{4}$ b) $5\frac{2}{3}$ c) $2\frac{5}{8}$ d) $5\frac{1}{6}$ e) $3\frac{3}{7}$

3.a) $\frac{11}{4}$ 4.a) $\frac{27}{5}$ b) $\frac{11}{3}$ c) $\frac{11}{7}$ d) $\frac{21}{4}$ e) $\frac{25}{9}$

Activity Exercise 2.2.

1. $\frac{23}{5} = \frac{20}{5} + \frac{3}{5}$
$= \frac{20 \div 5}{5 \div 5} + \frac{3}{5}$
$= \frac{4}{1} + \frac{3}{5}$
$= 4 + \frac{3}{5}$
$= 4\frac{3}{5}$

2.a) $\frac{37}{6} = \frac{36}{6} + \frac{1}{6}$
$= \frac{36 \div 6}{6 \div 6} + \frac{1}{6}$
$= \frac{6}{1} + \frac{1}{6}$
$= 6 + \frac{1}{6}$
$= 6\frac{1}{6}$

2.b) $\frac{17}{3} = \frac{15}{3} + \frac{2}{3}$
$= \frac{15 \div 3}{3 \div 3} + \frac{2}{3}$
$= \frac{5}{1} + \frac{2}{3}$
$= 5 + \frac{2}{3}$
$= 5\frac{2}{3}$

3. $3\frac{5}{7} = 3 + \frac{5}{7}$
$= \frac{3}{1} + \frac{5}{7}$
$= \frac{3 \times 7}{1 \times 7} + \frac{5}{7}$
$= \frac{21}{7} + \frac{5}{7}$
$= \frac{26}{7}$

4.a) $2\frac{3}{4} = 2 + \frac{3}{4}$
$= \frac{2}{1} + \frac{3}{4}$
$= \frac{2 \times 4}{1 \times 4} + \frac{3}{4}$
$= \frac{8}{4} + \frac{3}{4}$
$= \frac{11}{4}$

b) $6\frac{5}{8} = 6 + \frac{5}{8}$
$= \frac{6}{1} + \frac{5}{8}$
$= \frac{6 \times 8}{1 \times 8} + \frac{5}{8}$
$= \frac{48}{8} + \frac{5}{8}$
$= \frac{53}{8}$

CHECK EXERCISE 2A

B2.1 i) a) $2\frac{4}{15}$ b) $2\frac{14}{17}$ c) $2\frac{7}{10}$ d) $1\frac{3}{7}$ e) $3\frac{2}{5}$

ii) a) $\frac{29}{8}$ b) $\frac{47}{6}$ c) $\frac{52}{15}$ d) $\frac{23}{12}$ e) $\frac{125}{22}$

I2.2 i) a) $\frac{28}{3} = \frac{27}{3} + \frac{1}{3} = \frac{27 \div 3}{3 \div 3} + \frac{1}{3} = \frac{9}{1} + \frac{1}{3} = 9 + \frac{1}{3} = 9\frac{1}{3}$

b) $\frac{20}{9} = \frac{18}{9} + \frac{2}{9} = \frac{18 \div 9}{9 \div 9} + \frac{2}{9} = \frac{2}{1} + \frac{2}{9} = 2 + \frac{2}{9} = 2\frac{2}{9}$

ii) a) $8\frac{5}{6} = 8 + \frac{5}{6} = \frac{8}{1} + \frac{5}{6} = \frac{8 \times 6}{1 \times 6} + \frac{5}{6} = \frac{48}{6} + \frac{5}{6} = \frac{53}{6}$

b) $7\frac{3}{4} = 7 + \frac{3}{4} = \frac{7}{1} + \frac{3}{4} = \frac{7 \times 4}{1 \times 4} + \frac{3}{4} = \frac{28}{4} + \frac{3}{4} = \frac{31}{4}$

CHECK EXERCISE 3

I3.1 a) $8\frac{1}{40}$ b) $13\frac{9}{20}$ c) $9\frac{2}{15}$ d) $7\frac{2}{3}$ e) $17\frac{8}{15}$

I3.2 a) $4\frac{8}{15}$ b) $2\frac{5}{12}$ c) $12\frac{3}{8}$ d) $51\frac{39}{56}$ e) $7\frac{13}{18}$

Activity Exercise 3.1

1. $8\frac{5}{7}$ 2a) $8\frac{7}{10}$ b) $33\frac{19}{20}$ c) $15\frac{23}{24}$ d) $4\frac{3}{5}$ e) $11\frac{33}{70}$

Activity Exercise 3.2

1. $3\frac{3}{5}$ 2. $3\frac{4}{15}$ 3.a) $1\frac{1}{4}$ b) $1\frac{1}{16}$ c) $2\frac{7}{8}$ d) $\frac{15}{16}$ e) $2\frac{41}{70}$

CHECK EXERCISE 3A

I3.1 a) $13\frac{3}{5}$ b) $15\frac{19}{24}$ c) $12\frac{13}{36}$ d) $12\frac{43}{84}$ e) $10\frac{41}{60}$

I3.2 a) $1\frac{1}{4}$ b) $1\frac{11}{12}$ c) $\frac{13}{20}$ d) $\frac{1}{2}$ e) $\frac{11}{12}$

CHECK EXERCISE 4

I4.1 a) $p = \frac{1}{30}$ b) $a = \frac{5}{48}$ c) $n = \frac{1}{3}$ d) $b = 12\frac{11}{24}$ e) $a = 7\frac{27}{35}$

Activity Exercise 4.1

1.a) $n = 3\frac{1}{5}$ b) $b = 2\frac{1}{35}$ 2a) $a = 5\frac{11}{15}$ b) $b = 1\frac{7}{9}$

CHECK EXERCISE 4A

I4.1 a) $r = 1\frac{1}{8}$ b) $b = 4\frac{1}{3}$ c) $y = 6\frac{13}{20}$ d) $a = 15\frac{3}{10}$ e) $n = 5\frac{3}{16}$

CHECK EXERCISE 5

I5.1 1. 2 miles 2. $35\frac{7}{8}$ miles 3. $1\frac{11}{12}$ feet 4. $3\frac{1}{3}$ hours

Activity Exercise 5

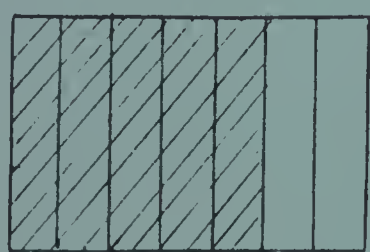
1. $\frac{103}{120}$ 2. $12\frac{1}{16}$ inches 3. $6\frac{1}{16}$ inches 4. $4\frac{85}{128}$ inches

CHECK EXERCISE 5A

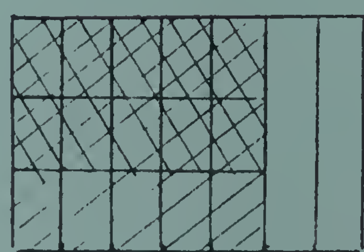
I5.1 1. $1\frac{1}{30}$ hours 2. $12\frac{1}{2}$ inches 3. $7\frac{15}{16}$ inches 4. up $2\frac{13}{24}$ points

CHECK EXERCISE 6

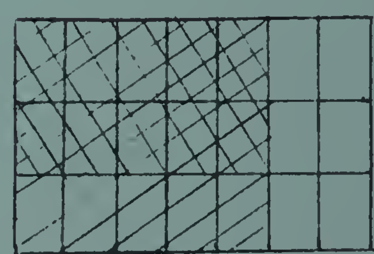
I6.1 i)



$\frac{5}{7}$ of region

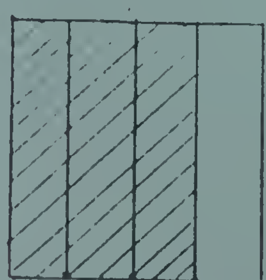


$\frac{2}{3}$ of $\frac{5}{7}$

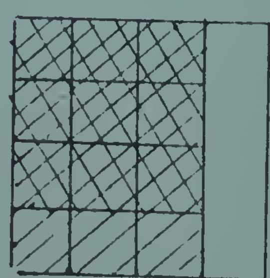


$\frac{2}{3}$ of $\frac{5}{7}$ is $\frac{10}{21}$

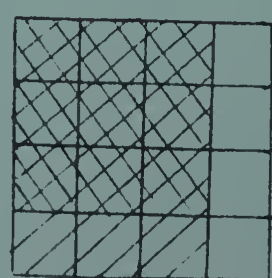
ii)



$\frac{3}{4}$ of region



$\frac{3}{4}$ of $\frac{3}{4}$



$\frac{3}{4}$ of $\frac{3}{4}$ is $\frac{9}{16}$

Review Exercises Topic II

Operations with rational numbers

You have completed the study of Topic II. It would be nice to know just how much you have learned in this topic. To do this you take a test, but before you take the test you should carefully review what you were expected to learn. To help you with this review we have prepared a set of exercises. If you have difficulty with any one of the exercises you should go back and review the appropriate objective (pink pages) and its development (white pages). The appropriate objective(s) for each exercise is (are) shown in the left margin next to the exercise.

I1.1 Find the sums and write each as a basic fraction:

A. $\frac{7}{9} + \frac{2}{5} + \frac{2}{3}$

B.
$$\begin{array}{r} \frac{7}{10} \\ \frac{1}{5} \\ \frac{5}{12} \\ +\ \frac{1}{2} \\ \hline \end{array}$$

I1.2 Find the differences for the following and write each as a basic fraction:

A. $\frac{9}{10} - \frac{3}{4}$

B.
$$\begin{array}{r} \frac{5}{12} \\ -\ \frac{3}{10} \\ \hline \end{array}$$

I2.1 A. Write a mixed numeral for each of the following fractions

i) $\frac{83}{6}$ ii) $\frac{257}{8}$

B. Write the following mixed numerals as fractions:

i) $6\frac{3}{5}$ ii) $15\frac{3}{8}$

I2.2 A. Justify that the mixed numeral for $\frac{43}{8}$ is $5\frac{3}{8}$.

B. Justify that the fraction for $7\frac{4}{5}$ is $\frac{39}{5}$.

I3.1 Find the following sums and write the fractional part as a basic fraction:

A. $3\frac{7}{8} + 2\frac{3}{4} + 1\frac{1}{2} + 6\frac{1}{4}$

B. $$\begin{array}{r} 32\frac{3}{8} \\ 16\frac{1}{3} \\ +\ 15\frac{5}{6} \\ \hline \end{array}$$

I3.2 Find the following differences and write the fractional part as a basic fraction:

A. $$\begin{array}{r} 8\frac{4}{7} \\ -\ 3\frac{5}{8} \\ \hline \end{array}$$

B. $4\frac{5}{12} - \frac{7}{9}$

I4.1 Solve each condition and write the fractions in the solutions as basic fractions:

A. $3\frac{5}{7} = n + 1\frac{3}{5}$

B. $n - \frac{3}{10} = 1\frac{5}{8}$

I5.1 John went on a hike. He covered a total distance of $2\frac{3}{16}$ miles. Yet when he came home he declared that he had walked only $1\frac{2}{3}$ miles. When questioned about this statement he said that he had run $\frac{25}{48}$ mile. Was John correct when he said that he had walked $1\frac{2}{3}$ miles?

I6.1 Show by diagrams how you can obtain $\frac{3}{4}$ of $\frac{4}{5}$.

I7.1 Find the basic fraction for each product:

A. $\frac{49}{66} \times \frac{27}{35} \times \frac{55}{63}$

B. $3 \times \frac{14}{75} \times 6\frac{3}{7}$

I8.1 i) Give an example that illustrates the property which a number and its reciprocal have.

ii) What is the reciprocal of 1?

iii) Write a whole number other than one and give its reciprocal.

iv) Give the rational number that has no reciprocal and explain why it has no reciprocal.

I9.1 Find the quotients of the following pairs of rational numbers:

A. $7\frac{5}{7} \div 3\frac{3}{14}$

B. $\dfrac{2\frac{1}{4}}{3\frac{3}{5}}$

I9.2 Explain in terms of reciprocals why $\frac{7}{12} \div 0$ is not possible. 178

I10.1 Solve each of the following conditions:

A. $6\frac{1}{4}\ a = 5\frac{5}{8}$ B. $1\frac{3}{4} = 2\frac{3}{16}\ n$

I11.1 A. A dealer bought a motor bike for \$1200. He added $\frac{1}{6}$ of this cost to determine his selling price. What was his selling price?

B. Two sides of a triangular flower bed are $1\frac{2}{3}$ yards and $2\frac{3}{4}$ yards long. The perimeter of the flower bed is 8 yards. What is the length of the third side?

C. An inch is about $2\frac{1}{2}$ centimeters. How many inches are there in $26\frac{1}{4}$ centimeters?

I12.1 Give a numerical example which illustrates what is meant by each of the following statements. (one example for each)

(a) the set of rational numbers is closed under addition.

(b) addition of rational numbers is commutative.

(c) addition of rational numbers is associative.

(d) the set of rational numbers has an identity element for addition.

I12.2 Show what is meant by each of the following statements by giving a numerical example for each.

(a) the set of rational numbers is closed under multiplication.

(b) multiplication of rational numbers is commutative.

(c) multiplication of rational numbers is associative.

(d) the set of rational numbers has an identity element for multiplication.

(e) rational numbers have the property that multiplication is distributive over addition.

(f) every non-zero rational number has a reciprocal.

I12.3 (a) State a property of multiplication of rational numbers which the whole numbers do not have for multiplication.

(b) Give an example to illustrate this property of the rational numbers and one to show that the whole numbers do not have it.

Answers to Review Exercises Topic II

I1.1 A. $\frac{83}{45}$ B. $\frac{109}{60}$

I1.2 A. $\frac{3}{20}$ B. $\frac{7}{60}$

B2.1 A. i) $13\frac{5}{6}$; ii) $32\frac{1}{8}$ B. i) $\frac{33}{5}$; ii) $\frac{123}{8}$

I2.2 A.

$$\begin{aligned}\frac{43}{8} &= \frac{40}{8} + \frac{3}{8}\\ &= \frac{40 \div 8}{8 \div 8} + \frac{3}{8}\\ &= \frac{5}{1} + \frac{3}{8}\\ &= 5 + \frac{3}{8}\\ &= 5\frac{3}{8}\end{aligned}$$

B.

$$\begin{aligned}7\frac{4}{5} &= 7 + \frac{4}{5}\\ &= \frac{7}{1} + \frac{4}{5}\\ &= \frac{7 \times 5}{1 \times 5} + \frac{4}{5}\\ &= \frac{35}{5} + \frac{4}{5}\\ &= \frac{39}{5}\end{aligned}$$

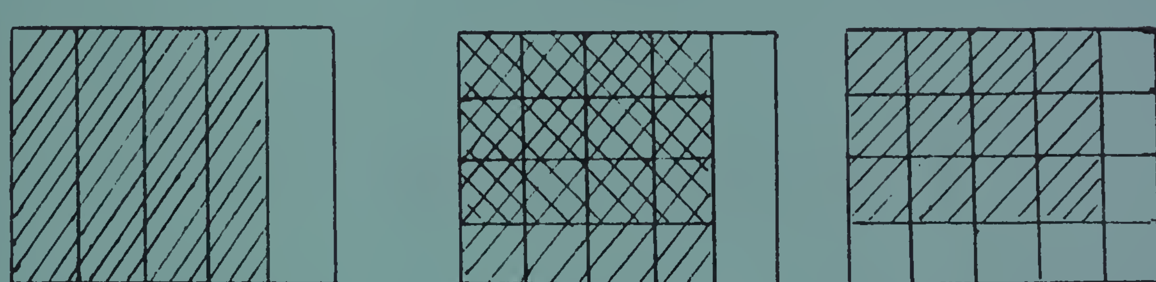
I3.1 A. $14\frac{3}{8}$ B. $64\frac{13}{24}$

I3.2 A. $4\frac{53}{56}$ B. $3\frac{23}{36}$

I4.1 A. $n = 2\frac{4}{35}$ B. $n = 1\frac{37}{40}$

I5.1 Yes

I6.1



$\frac{4}{5}$ $\quad$ $\frac{3}{4}$ of $\frac{4}{5}$ $\quad$ $\frac{3}{4}$ of $\frac{4}{5} = \frac{12}{20}$

I7.1 A. $\frac{1}{2}$ B. $\frac{18}{5}$ or $3\frac{3}{5}$

I8.1 i) $\frac{3}{4} \times \frac{4}{3} = 1$ any fraction can be used to illustrate this property

ii) 1

iii) can be any whole number, i.e. $17, \frac{1}{17}$.

iv) 0 has no reciprocal; $0 = \frac{0}{1}$ and there is no number by which $\frac{0}{1}$ can be multiplied to give 1, for $\frac{0}{1}$ times any number is 0.

OR. The reciprocal of $\frac{0}{1}$ is $\frac{1}{0}$, but $\frac{1}{0}$ has no meaning since we cannot divide by 0.

I9.1 A. $\frac{12}{5} = 2\frac{2}{5}$ B. $\frac{5}{8}$

I9.2 $\frac{7}{12} \div 0$ can be solved by converting to the corresponding multiplication and get $\frac{7}{12} \times \frac{1}{0}$, but $\frac{1}{0}$ does not exist and hence we cannot get the multiplication example. Therefore the division is not possible.

I10.1 A. $a = \frac{9}{10}$ B. $n = \frac{4}{5}$

I11.1 A. selling price is \$1400 ; B. $3\frac{7}{12}$ yards; C. $10\frac{1}{2}$ inches

I12.1 Examples which show the same ideas as:

a) $\frac{3}{5} + \frac{5}{7} = \frac{46}{35}$ and $\frac{46}{35}$ is a rational number.

b) $\frac{3}{5} + \frac{5}{7} = \frac{5}{7} + \frac{3}{5}$

c) $(\frac{3}{5} + \frac{5}{7}) + \frac{7}{8} = \frac{3}{5} + (\frac{5}{7} + \frac{7}{8})$

d) $\frac{3}{5} + 0 = \frac{3}{5}$

I12.2 Examples which show the same ideas as:

a) $\frac{3}{5} \times \frac{5}{7} = \frac{3}{7}$ and $\frac{3}{7}$ is a rational number.

b) $\frac{3}{5} \times \frac{5}{7} = \frac{5}{7} \times \frac{3}{5}$

c) $(\frac{3}{5} \times \frac{5}{7}) \times \frac{7}{9} = \frac{3}{5} \times (\frac{5}{7} \times \frac{7}{9})$

d) $\frac{3}{5} \times 1 = \frac{3}{5}$

e) $\frac{3}{5} \times (\frac{5}{7} + \frac{7}{9}) = (\frac{3}{5} \times \frac{5}{7}) + (\frac{3}{5} \times \frac{7}{9})$

f) reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.

I12.3 a) A statement which gives the same idea as: Every non-zero rational number has a reciprocal.

b) Examples which show the same idea as:

$\frac{3}{5}$ and $\frac{5}{3}$ are reciprocals for $\frac{3}{5} \times \frac{5}{3} = 1$.

3 has no reciprocal for there exists no whole number by which you can multiply 3 and get one as an answer.

TOPIC II (OPERATIONS WITH RATIONAL NUMBERS)

PHASE II

BASIC LEVEL

Students who achieved less than half of the objectives in Phase I are given this material. It is prepared to let you work at the level of difficulty which suits you. Using it will help you prepare for the next test.

On the next test, you will be expected to answer only those questions which relate to BASIC objectives that you did not achieve on the test you have just written. If you achieved an INTERMEDIATE objective, you will not have to work on the basic objective with the same number, (For example, if you got the question relating to objective I3.2 correct on the first test, then you have already achieved objective B3.2).

Use your record page to tell you which objectives you have not yet achieved. Use your new flow chart to guide you through Phase II and to keep a record of what you have done. Use the objectives in phase I materials to help you know what you have to learn. Use the following activities and exercises to practice your skills for the objectives you have not achieved.

BASIC LEVEL

OBJECTIVE B1

For each of the objectives in section 1 that you did not achieve do the following:

- Read the objective and the corresponding description.
- Do the appropriate exercises and CHECK EXERCISES on these pages.

B1.1 Addition of rational numbers named by fractions.

$\frac{5}{3} + \frac{1}{2} + \frac{4}{9}$

To do the addition at the left:

(1) Find the least (smallest) number which 3, 2 and 9 divide into; i.e. the least common multiple of 3, 2 and 9. It is 18. This is the least common denominator. (L.C.D.)

(2) For each fraction, find an equivalent fraction with the common denominator of 18.
$\frac{5}{3} = \frac{5 \times 6}{3 \times 6} = \frac{30}{18}$; $\frac{1}{2} = \frac{1 \times 9}{2 \times 9} = \frac{9}{18}$; $\frac{4}{9} = \frac{4 \times 2}{9 \times 2} = \frac{8}{18}$

$= \frac{30}{18} + \frac{9}{18} + \frac{8}{18}$

(3) Add the numerators.

$= \frac{47}{18}$

Find the following sums, and write each as a basic fraction:

1. $\frac{3}{4} + \frac{5}{6}$ (remember to find the least common denominator for each fraction and to form equivalent fractions with this L.C.D.)
2. $\frac{1}{3} + \frac{7}{9} + \frac{5}{6}$ (You must find the L.C.D. for all three fractions).
3. $\frac{3}{8} + \frac{1}{6} + \frac{5}{12}$
4. $\frac{2}{3} + \frac{1}{4} + \frac{5}{9}$
5. $\frac{3}{7} + \frac{1}{2} + \frac{5}{4}$

6.
$$\begin{array}{rcl} \frac{3}{5} & = & \boxed{\frac{\ }{\ }} \\ \frac{1}{2} & = & \boxed{\frac{\ }{\ }} \\ +\ \frac{3}{4} & = & +\boxed{\frac{\ }{\ }} \\ \hline \end{array}$$

7.
$$\begin{array}{r} \frac{7}{8} \\ \frac{3}{4} \\ +\ \frac{5}{16} \\ \hline \end{array}$$

B1.2 Subtraction of rational numbers named by fractions.

$\frac{5}{6} - \frac{3}{10}$

To do the subtraction at the left:

(1) Find the least (smallest) number which 6 and 10 divide into; i.e. the least common multiple of 6 and 10. It is 30. This is the least common denominator.

(2) For each fraction, find an equivalent fraction with the common denominator of 30.
$\frac{5}{6} = \frac{5 \times 5}{6 \times 5} = \frac{25}{30}$; $\frac{3}{10} = \frac{3 \times 3}{10 \times 3} = \frac{9}{30}$

$= \frac{25}{30} - \frac{9}{30}$

$= \frac{16}{30}$ (3) Subtract the numerators.

$= \frac{8}{15}$ (4) Give the answer as a basic fraction.

Find the differences, and write each as a basic fraction:

1. $\frac{3}{4} - \frac{1}{3}$ (remember to find the least common denominator for each fraction and to form equivalent fractions each with this denominator.)
2. $\frac{4}{5} - \frac{3}{4}$ (Again the L.C.D. must be found before subtracting).
3. $\frac{5}{8} - \frac{5}{12}$
4. $\frac{5}{6} - \frac{5}{8}$
5. $\frac{8}{9} - \frac{5}{6}$
6. $\frac{5}{8} = \frac{\square}{\square}$, $-\frac{1}{2} = \frac{\square}{\square}$
7. $\frac{11}{8} - \frac{3}{4}$

- Check your answers with those given at the end of the topic.
- Do the required parts in CHECK EXERCISE B1.

CHECK EXERCISE B1

B1.1 Find the sums and write each as a basic fraction:

a) $\frac{3}{4} + \frac{7}{8} + \frac{1}{2}$

b) $\frac{5}{2} + \frac{3}{4} + \frac{4}{5}$

c) $\frac{3}{8} + \frac{3}{4} + \frac{2}{3}$

d) $\frac{1}{2} + \frac{3}{8} + \frac{3}{4}$

e) $\frac{3}{4} + \frac{1}{12} + \frac{11}{6}$

B1.2 Find the differences and write each as a basic fraction:

a) $\frac{3}{4} - \frac{2}{3}$

b) $\frac{3}{2} - \frac{5}{8}$

c) $\frac{5}{9} - \frac{1}{2}$

d) $\frac{7}{6} - \frac{3}{4}$

e) $\frac{5}{2} - \frac{3}{5}$

- Check your answers with those given at the end of the topic.

  You were successful on CHECK EXERCISE

  B1.1 if you had at least 4 of the 5 parts correct.

  B1.2 if you had at least 4 of the 5 parts correct.

- If you are not certain how to add or subtract rational numbers, or if you were unsuccessful on a CHECK EXERCISE, consult your teacher. Then do the appropriate exercises that follow.
- Otherwise, go on to your next unachieved objective.

EXERCISES B1

B1.1 Find the sums and write each as a basic fraction:

a) $\frac{1}{2} + \frac{5}{6} + \frac{7}{12}$

b) $\frac{3}{8} + \frac{5}{12} + \frac{3}{4}$

c) $\frac{9}{16} + \frac{5}{8} + \frac{1}{2}$

d) $\begin{array}{r} \frac{1}{4} \\ \frac{3}{5} \\ + \frac{3}{4} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{5}{8} \\ \frac{5}{6} \\ + \frac{2}{3} \\ \hline \end{array}$

B1.2 Find the differences and write each as a basic fraction:

a) $\frac{4}{5} - \frac{2}{5}$

b) $\frac{5}{8} - \frac{1}{2}$

c) $\frac{3}{5} - \frac{1}{4}$

d) $\begin{array}{r} \frac{3}{4} \\ - \frac{2}{3} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{5}{6} \\ - \frac{3}{4} \\ \hline \end{array}$

- Check your answers with those given at the end of the topic.
- Go on to your next unachieved objective.

OBJECTIVE B2

- Read objective B2.1 and its description in section 2.
- Do the following exercises.

1. Write the following fractions as mixed numerals:

   a) $\frac{10}{3}$ (remember: $\frac{10}{3} = \frac{9}{3} + \frac{1}{3} = 3 + \frac{1}{3} =$ ______)
   or $\begin{array}{r} 3 \\ 3\overline{)10} \end{array}$ i.e. $3\frac{1}{3}$

   b) $\frac{11}{3}$ (remember: $\frac{11}{3} = \frac{9}{3} + \frac{2}{3} =$ ____ $=$ ____)
   or $\begin{array}{r} 3 \text{ rem. } 2 \\ 3\overline{)11} \end{array}$ i.e. $3\frac{2}{3}$

   c) $\frac{11}{8}$ f) $\frac{15}{4}$

   d) $\frac{22}{7}$ g) $\frac{33}{10}$

   e) $\frac{5}{2}$

2. Write the following mixed numerals as fractions:

   a) $3\frac{1}{2}$ (remember: $3\frac{1}{2} = 3 + \frac{1}{2} = \frac{6}{2} + \frac{1}{2} =$ ______
   or $3\frac{1}{2} = 3 + \frac{1}{2} =$ 6 halves + 1 half $= \frac{7}{2}$)

   b) $2\frac{4}{5}$ (remember: $2\frac{4}{5} = 2 + \frac{4}{5} = \frac{10}{5} + \frac{—}{} =$ ____
   or $2\frac{4}{5} = 2 + \frac{4}{5} =$ 10 fifths + 4 fifths = 14 fifths $= \frac{14}{5}$

   c) $1\frac{2}{3}$ f) $3\frac{3}{7}$

   d) $6\frac{2}{3}$ g) $1\frac{3}{8}$

   e) $7\frac{1}{2}$

- Check your answers with those given at the end of the topic.
- Do CHECK EXERCISE B2.

CHECK EXERCISE B2

B2.1 i) Write the following mixed numerals as fractions:

a) $3\frac{2}{5}$ d) $4\frac{4}{5}$

b) $1\frac{5}{8}$ e) $3\frac{1}{4}$

c) $2\frac{1}{2}$

B2.1 ii) Write the following fractions as mixed numerals:

a) $\frac{11}{4}$ d) $\frac{13}{3}$

b) $\frac{15}{8}$ e) $\frac{17}{8}$

c) $\frac{25}{4}$

- Check your answers with those given at the end of the topic.
- If you are unsure how to do the above exercises or if you had more than one part incorrect in either question (i) or question (ii); consult your teacher. Then do the following exercises.
- Otherwise, go on to your next unachieved objective.

EXERCISES B2

B2.1 i) Write the following mixed numerals as fractions:

a) $3\frac{1}{2}$ d) $4\frac{1}{4}$

b) $5\frac{3}{10}$ e) $1\frac{5}{6}$

c) $1\frac{3}{10}$

ii) Write the following fractions as mixed numerals:

a) $\frac{25}{8}$ d) $\frac{19}{7}$

b) $\frac{11}{6}$ e) $\frac{14}{5}$

c) $\frac{8}{3}$

- Check your answers with those given at the end of the topic.
- Go on to your next unachieved objective.

OBJECTIVE B3

For each of the objectives in section 3 that you did not achieve, do the following:

- Read the objective and corresponding description.
- Do the appropriate exercises and CHECK EXERCISES.

B3.1 Find the sums and write each as a mixed numeral with fractional part as a basic fraction:

a) $\begin{array}{r} 7\frac{1}{4} \\ +\ 2\frac{1}{2} \\ \hline \end{array}$ (Remember: $\begin{array}{rcl} 7\frac{1}{4} & = & 7 + \frac{1}{4} \\ +\ 2\frac{1}{2} & = & 2 + \frac{1}{2} \\ \hline & & 9 + \frac{3}{4} = \underline{\qquad} \end{array}$)

b) $7\frac{5}{6} + 4\frac{2}{3}$ (Remember: $7\frac{5}{6} + 4\frac{2}{3} = 7 + \frac{5}{6} + 4 + \frac{4}{6}$
$= 11 + \frac{9}{6}$
$= 11 + \underline{\quad}$
$= \underline{\qquad}$)

c) $6\frac{1}{12} + 5\frac{1}{6} + 9\frac{2}{3}$

d) $8\frac{3}{5} + 13\frac{3}{4} + 3\frac{7}{10}$

e) $7\frac{1}{3} + 5\frac{3}{5} + 2\frac{1}{15}$

f) $\begin{array}{r} 4\frac{7}{12} \\ 9\ \frac{1}{2} \\ +\ 17\ \frac{3}{8} \\ \hline \end{array}$

g) $\begin{array}{r} 4\frac{5}{16} \\ 14\ \frac{1}{2} \\ +\ 3\ \frac{3}{8} \\ \hline \end{array}$

B3.2 Find the differences and write each as a mixed numeral with fractional part as a basic fraction.

a) $\begin{array}{rcl} 5\frac{3}{4} & = & 5 + \frac{9}{12} \\ -\ 2\frac{2}{3} & = & -2 + \underline{\quad} \\ \hline & & 3 + \underline{\quad} = \underline{\quad} \end{array}$

b) $5\frac{1}{4} - 2\frac{3}{4} = \underset{(5-2)}{3} + \frac{1}{4} - \frac{3}{4}$
$= 2 + 1\frac{1}{4} - \frac{3}{4}$
$= 2 + \underline{\quad} - \frac{3}{4}$
$= 2 + \underline{\quad}$
$= 2\frac{1}{2}$

c) $\begin{array}{r} 6\frac{3}{8} \\ -4\frac{3}{4} \\ \hline \end{array}$ d) $\begin{array}{r} 14\frac{4}{5} \\ -\ 13\frac{1}{2} \\ \hline \end{array}$ e) $19\frac{1}{2} - 12\frac{5}{7}$

f) $9 - 5\frac{2}{3}$

- Check your answers with those given at the end of the topic.
- Do the required parts in CHECK EXERCISE B3.

CHECK EXERCISE B3

B3.1 Find the sums and write each as a mixed numeral with fractional part as a basic fraction:

a) $6\frac{2}{3} + 4\frac{3}{4} + 5\frac{5}{6}$

b) $10\frac{3}{4} + 4\frac{1}{12} + 2\frac{5}{6}$

c) $7\frac{4}{5} + 8\frac{3}{4} + 2\frac{3}{10}$

d) $\begin{array}{r} 17\frac{1}{3} \\ 5\frac{5}{6} \\ +\ 4\frac{1}{2} \\ \hline \end{array}$ e) $\begin{array}{r} 3\ \frac{1}{8} \\ 2\frac{1}{16} \\ +\ 1\ \frac{1}{2} \\ \hline \end{array}$

B3.2 Find the differences and write each as a mixed numeral with fractional part as a basic fraction:

a) $5\frac{1}{2} - 3\frac{3}{8}$

b) $4\frac{3}{4} - 2\frac{1}{6}$

c) $17\frac{3}{4} - 6\frac{7}{8}$

d) $\begin{array}{r} 9\frac{1}{2} \\ -7\frac{5}{8} \\ \hline \end{array}$ e) $\begin{array}{r} 3\ \frac{3}{5} \\ -1\frac{9}{10} \\ \hline \end{array}$

- Check your answers with those given at the end of the topic.
  You were successful on CHECK EXERCISE
  B3.1 if you had at least 4 of the parts correct,
  B3.2 if you had at least 4 of the parts correct.
- If you are unsure how to add or subtract with mixed numerals, or if you were unsuccessful on either CHECK EXERCISE; consult your teacher. Then do the appropriate exercise that follows.
- Otherwise, go on to your next unachieved objective.

EXERCISES B3

B3.1 Find the sums and write each as a mixed numeral with fractional part as a basic fraction.

a) $3\frac{1}{12} + 5\frac{1}{6} + 7\frac{2}{3}$

b) $8\frac{1}{2} + 7\frac{3}{4} + 12\frac{5}{8}$

c) $7\frac{2}{3} + 11\frac{3}{4} + 8\frac{5}{9}$

d)
$$\begin{array}{r} 11\frac{5}{9} \\ 8\frac{1}{3} \\ +\ 7\frac{5}{6} \\ \hline \end{array}$$

e)
$$\begin{array}{r} 22\ \frac{2}{9} \\ 6\frac{5}{12} \\ +\ 4\ \frac{1}{4} \\ \hline \end{array}$$

B3.2 Find the differences and write each as a mixed numeral with fractional part as a basic fraction.

a) $6\frac{2}{3} - 2\frac{3}{8}$

b) $8\frac{1}{4} - 3\frac{2}{3}$

c) $15\frac{3}{8} - 9\frac{5}{12}$

d)
$$\begin{array}{r} 7\frac{1}{2} \\ -\ 3\frac{2}{5} \\ \hline \end{array}$$

e)
$$\begin{array}{r} 5\frac{1}{4} \\ -\ 2\frac{5}{6} \\ \hline \end{array}$$

- Check your answers with those given at the end of the topic.
- Go on to your next unachieved objective.

OBJECTIVE B4

- Read objective B4.1 and the corresponding description.
- Do the following exercises.

B4.1 Solution of conditions for equality in which the universe or replacement set for the variable is the set of rational numbers.

$3\frac{4}{5} + n = 5\frac{1}{3}$ To solve the condition at the left:

$n = 5\frac{1}{3} - 3\frac{4}{5}$ (1) form the corresponding condition involving subtraction.

$= 2 + \frac{5}{15} - \frac{12}{15}$ (2) do the subtraction.

$= 1 + \frac{20}{15} - \frac{12}{15}$

$= 1\frac{8}{15}$

Check: $3\frac{4}{5} + 1\frac{8}{15}$ (3) check the solution by replacing n.

$= 4 + \frac{12}{15} + \frac{8}{15}$

$= 4 + \frac{20}{15}$

$= 5\frac{5}{15}$

$= 5\frac{1}{3}$

Solution: $1\frac{8}{15}$ (4) Give the solution.

Solve the following conditions of equality and write fractions in solutions as basic fractions. The universe or replacement set for each variable is the set of rational numbers.

a) $7\frac{4}{5} + n = 8\frac{3}{4}$

$n = 8\frac{3}{4} - 7\frac{4}{5}$ form the corresponding condition involving subtraction.

$= 1 + \frac{15}{20} + \boxed{\frac{\quad}{\quad}}$

$= \frac{\boxed{}}{20} - \frac{16}{20}$

$n = \boxed{\frac{\quad}{\quad}}$

Check:

Solution:

OBJECTIVE B1

B1.1 1. $\frac{19}{12}$ 2. $\frac{35}{18}$ 3. $\frac{23}{24}$ 4. $\frac{53}{36}$ 5. $\frac{61}{28}$ 6. $\frac{12}{20}, \frac{10}{20}, \frac{15}{20}; \frac{37}{20}$
7. $\frac{31}{16}$

B1.2 1. $\frac{5}{12}$ 2. $\frac{1}{20}$ 3. $\frac{5}{24}$ 4. $\frac{5}{24}$ 5. $\frac{1}{18}$ 6. $\frac{5}{8}, \frac{4}{8}; \frac{1}{8}$ 7. $\frac{5}{8}$

CHECK EXERCISE B1

B1.1 a) $\frac{17}{8}$ b) $\frac{81}{20}$ c) $\frac{43}{24}$ d) $\frac{13}{8}$ e) $\frac{8}{3}$

B1.2 a) $\frac{1}{12}$ b) $\frac{7}{8}$ c) $\frac{1}{18}$ d) $\frac{5}{12}$ e) $\frac{19}{10}$

EXERCISES B.1

B1.1 a) $\frac{23}{12}$ b) $\frac{37}{24}$ c) $\frac{27}{16}$ d) $\frac{8}{5}$ e) $\frac{51}{24}$

B1.2 a) $\frac{2}{5}$ b) $\frac{1}{8}$ c) $\frac{7}{20}$ d) $\frac{1}{12}$ e) $\frac{1}{12}$

OBJECTIVE B2

B2.1 1. a) $3\frac{1}{3}$ b) $3\frac{2}{3}$ c) $1\frac{3}{8}$ d) $3\frac{1}{7}$ e) $2\frac{1}{2}$ f) $3\frac{3}{4}$ g) $3\frac{3}{10}$
2. a) $\frac{7}{2}$ b) $\frac{14}{5}$ c) $\frac{5}{3}$ d) $\frac{20}{3}$ e) $\frac{15}{2}$ f) $\frac{24}{7}$ g) $\frac{11}{8}$

CHECK EXERCISE B2

B2.1 i) a) $\frac{17}{5}$ b) $\frac{13}{8}$ c) $\frac{5}{2}$ d) $\frac{24}{5}$ e) $\frac{13}{4}$
ii) a) $2\frac{3}{4}$ b) $1\frac{7}{8}$ c) $6\frac{1}{4}$ d) $4\frac{1}{3}$ e) $2\frac{1}{8}$

EXERCISES B2

B2.1 i) a) $\frac{7}{2}$ b) $\frac{53}{10}$ c) $\frac{13}{10}$ d) $\frac{17}{4}$ e) $\frac{11}{6}$
ii) a) $3\frac{1}{8}$ b) $1\frac{5}{6}$ c) $2\frac{2}{3}$ d) $2\frac{5}{7}$ e) $2\frac{4}{5}$

OBJECTIVE B3

B3.1 a) $9\frac{3}{4}$ b) $12\frac{1}{2}$ c) $20\frac{11}{12}$ d) $26\frac{1}{20}$ e) 15 f) $31\frac{11}{24}$ g) $22\frac{3}{16}$

B3.2 a) $\frac{8}{12}, \frac{1}{12}, 3\frac{1}{12}$ b) $\frac{5}{4}, \frac{2}{4}$ c) $1\frac{5}{8}$ d) $1\frac{3}{10}$ e) $6\frac{11}{14}$ f) $3\frac{1}{3}$

CHECK EXERCISE B3

B3.1 a) $17\frac{1}{4}$ b) $17\frac{2}{3}$ c) $18\frac{17}{20}$ d) $27\frac{2}{3}$ e) $6\frac{11}{16}$

B3.2 a) $2\frac{1}{8}$ b) $2\frac{7}{12}$ c) $10\frac{7}{8}$ d) $1\frac{7}{8}$ e) $1\frac{7}{10}$

2B28

EXERCISES B3

B3.1 a) $15\frac{11}{12}$ b) $28\frac{7}{8}$ c) $27\frac{35}{36}$ d) $27\frac{13}{18}$ e) $32\frac{8}{9}$

B3.2 a) $4\frac{7}{24}$ b) $4\frac{7}{12}$ c) $5\frac{23}{24}$ d) $4\frac{1}{10}$ e) $2\frac{5}{12}$

OBJECTIVE B4

B4.1 a) $\frac{16}{20}$, 35; $n = \frac{19}{20}$ b) $2\frac{2}{3} + 6\frac{1}{6}$, $\frac{4}{6} + \frac{1}{6}$; $n = 8\frac{5}{6}$ c) $n = 2\frac{3}{8}$

d) $n = 24\frac{5}{8}$ e) $n = 3\frac{11}{30}$ f) $n = 14\frac{1}{8}$

CHECK EXERCISE B4

B4.1 a) $n = 16\frac{11}{20}$ b) $n = 1\frac{1}{2}$ c) $n = 7\frac{1}{12}$ d) $n = 2\frac{2}{15}$ e) $n = 5\frac{1}{2}$

EXERCISES B4

B4.1 a) $n = 7\frac{6}{7}$ b) $n = 29\frac{1}{4}$ c) $n = 2\frac{13}{20}$ d) $n = 1\frac{3}{8}$ e) $n = 14\frac{1}{5}$

OBJECTIVE B5

B5.1 a) $n = 3\frac{5}{16} + 5\frac{13}{16} + \frac{1}{16}$, $\frac{5}{16} + \frac{13}{16} + \frac{1}{16}$, $\frac{19}{16}$, $9\frac{3}{16}$; $9\frac{3}{16}$ inches.

b) $497\frac{29}{72}$lbs. c) $4\frac{7}{10}$ gal. d) $6\frac{5}{8}$ hr. e) $\frac{3}{8}$ inch f) $2\frac{7}{8}$ oz.

CHECK EXERCISE B5

B5.1 a) $14\frac{3}{8}$ lb. b) $6\frac{1}{8}$ yd. c) $10\frac{3}{8}$ in. d) $9\frac{1}{4}$ in. e) $\frac{1}{2}$ mi.

EXERCISES B5

B5.1 a) $\frac{5}{8}$ mi. b) $3\frac{1}{8}$ lb. c) $\frac{3}{8}$ lb. d) $1\frac{11}{12}$ lb. e) $3\frac{1}{10}$

OBJECTIVE B7

B7.1 a) $\frac{1}{5}$ b) $\frac{2 \times 9}{3 \times 1}$; 6 c) $\frac{13 \times 3}{9 \times 13}$; $\frac{1}{3}$ d) $\frac{3 \times 16}{4 \times 9}$; $1\frac{1}{3}$

e) 6 f) 33 g) $19\frac{4}{5}$ h) $\frac{1}{8}$ i) $9\frac{1}{6}$ j) 21

CHECK EXERCISE B7

B7.1 a) $\frac{7}{12}$ b) $7\frac{1}{2}$ c) 3 d) 45 e) 36

TOPIC II (OPERATIONS WITH RATIONAL NUMBERS)

PHASE II

INTERMEDIATE LEVEL

Students who achieved between half and nine tenths (90%) of the objectives in Phase I are given these materials.

Your job now is to master all of the objectives that you have missed.

On the next test, you will be expected to answer only those questions which are related to objectives that you did not achieve on the first test.

Use your record page to tell you which objectives you need to work on.

Use your flow chart to guide you through Phase II and to show what you have done in Phase I. You may want to do some of the parts you left out during Phase I.

Use your phase I materials to relearn the ideas for objectives you have not achieved and to give you examples of the type of questions you need to be able to answer.

Use these exercises to practice your skills so that you will be able to achieve all of the objectives.

INTERMEDIATE LEVEL

For each of the objectives you did not achieve on Post Test I, do the appropriate exercises and check your answers with those given at the end of the topic. Review the appropriate material in Phase I before starting each exercise.

OBJECTIVE I1.1

Find the following sums and write each as a basic fraction:

a) $\frac{5}{6} + \frac{3}{4} + \frac{4}{5}$

b) $\frac{3}{7} + \frac{1}{2} + \frac{2}{3}$

c) $\frac{5}{8} + \frac{5}{6} + \frac{5}{9}$

d) $\begin{array}{r} \frac{4}{9} \\ \frac{5}{12} \\ +\ \frac{2}{15} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{4}{15} \\ \frac{3}{20} \\ +\ \frac{5}{6} \\ \hline \end{array}$

OBJECTIVE I1.2

Find the differences and write each as a basic fraction:

a) $\frac{8}{9} - \frac{3}{5}$

b) $\frac{4}{11} - \frac{1}{7}$

c) $\frac{9}{16} - \frac{3}{20}$

d) $\begin{array}{r} \frac{14}{25} \\ -\ \frac{4}{15} \\ \hline \end{array}$

e) $\begin{array}{r} \frac{17}{24} \\ -\ \frac{3}{16} \\ \hline \end{array}$

OBJECTIVE B2.1

i) Write the following mixed numerals as fractions:

a) $3\frac{1}{2}$

b) $5\frac{3}{10}$

c) $1\frac{3}{8}$

d) $4\frac{1}{4}$

e) $7\frac{5}{6}$

ii) Write the following fractions as mixed numerals:

a) $\frac{25}{8}$

b) $\frac{11}{6}$

c) $\frac{28}{3}$

d) $\frac{19}{7}$

e) $\frac{34}{5}$

OBJECTIVE I5.1

Solve the following applied problems:

a) Carol worked as a babysitter one summer. During one week she worked $3\frac{1}{2}$ hours on Monday, $4\frac{2}{3}$ hours on Tuesday, $2\frac{4}{5}$ hours on Wednesday, $5\frac{1}{6}$ hours on Saturday and $3\frac{3}{10}$ hours on Sunday. How many hours did Carol work during the week?

b) Four sections of a highway totaling $15\frac{2}{3}$ miles are to be built. Three of the sections measure $3\frac{1}{2}$, $5\frac{3}{5}$ and $5\frac{1}{6}$ miles. What is the length of the fourth section?

c) John, who weighed $135\frac{3}{4}$ pounds, went on a diet. After the first week he lost $2\frac{2}{3}$ pounds. However, during the second week he gained $1\frac{2}{5}$ pounds. How much did he weigh at the end of the second week?

d) A group of scouts joined short lengths of rope together in order to descend a steep bank. The lengths measured $3\frac{1}{2}$ ft., $2\frac{2}{3}$ ft. 18 inches, and $4\frac{3}{8}$ ft. How many feet long was the joined rope? (Allow 1 foot for the knots.)

e) George and Bill leave their respective homes, $3\frac{2}{5}$ miles apart, planning to meet half-way. George walked $\frac{3}{4}$ of a mile then stopped to chat with a friend. He was still talking when Bill stopped for a bottle of pop $1\frac{1}{8}$ miles from his home. How far apart were the boys when they both stopped?

OBJECTIVE I6.1

a) Show by diagrams how $\frac{2}{3}$ of $\frac{3}{4}$ can be obtained.

b) Show by diagrams how $\frac{4}{5}$ of $\frac{2}{3}$ can be obtained.

OBJECTIVE I7.1

Find the following products:

a) $\frac{8}{9}$ of $7\frac{7}{8}$

b) $\frac{7}{8}$ of $\frac{4}{21}$

c) $\frac{14}{19} \times 3\frac{1}{7} \times 2\frac{3}{8}$

d) $\frac{4}{15} \times 22\frac{1}{2} \times 2\frac{2}{3}$

e) $2\frac{1}{4} \times 3\frac{1}{7} \times \frac{4}{27} \times 3$

OBJECTIVE I2.2

i) a) Use fractions to justify that the mixed numeral for $\frac{33}{7}$ is $4\frac{5}{7}$.

b) Use fractions to justify that the mixed numeral for $\frac{25}{3}$ is $8\frac{1}{3}$.

ii) a) Use fractions to justify that the fraction for $5\frac{2}{7}$ is $\frac{37}{7}$.

b) Use fractions to justify that the fraction for $6\frac{3}{7}$ is $\frac{45}{7}$.

OBJECTIVE I3.1

Find the following sums and write each as a mixed numeral with fractional part a basic fraction:

a) $9\frac{2}{3} + 13\frac{3}{4} + 7\frac{4}{5}$

b) $12\frac{7}{10} + 19\frac{3}{4} + 7\frac{7}{12}$

c) $4\frac{5}{16} + 17\frac{5}{9} + 10\frac{5}{12}$

d) $\begin{array}{r} 6\frac{7}{10} \\ 12\frac{4}{15} \\ +\ 2\frac{5}{6} \\ \hline \end{array}$

e) $\begin{array}{r} 11\frac{7}{9} \\ \frac{5}{6} \\ +\ 4\frac{3}{4} \\ \hline \end{array}$

OBJECTIVE I3.2

Find the differences and write the fraction parts of the mixed numerals as basic fractions:

a) $5\frac{7}{12} - 3\frac{8}{9}$

b) $11\frac{7}{9} - 3\frac{5}{6}$

c) $7 - 4\frac{2}{5}$

d) $\begin{array}{r} 8\frac{5}{24} \\ -\ 3\frac{7}{16} \\ \hline \end{array}$

e) $\begin{array}{r} 12\frac{11}{18} \\ -\ \frac{11}{12} \\ \hline \end{array}$

OBJECTIVE I4.1

Solve the following conditions of equality. The universe or replacement set for each variable is the set of rational numbers.

a) $3\frac{7}{9} = n + 2\frac{3}{4}$

b) $1\frac{1}{6} + p = 4\frac{1}{8}$

c) $m - 4\frac{5}{16} = 17\frac{5}{9}$

d) $12\frac{7}{15} = a - 9\frac{8}{9}$

e) $6\frac{3}{4} = 7\frac{2}{3} - z$

Answers

OBJECTIVE I1.1

a) $\frac{143}{60}$ b) $\frac{67}{42}$ c) $\frac{145}{72}$ d) $\frac{179}{180}$ e) $\frac{5}{4}$

OBJECTIVE I1.2

a) $\frac{13}{45}$ b) $\frac{17}{77}$ c) $\frac{33}{80}$ d) $\frac{22}{75}$ e) $\frac{25}{48}$

OBJECTIVE B2.1

i) a) $\frac{7}{2}$ b) $\frac{53}{10}$ c) $\frac{11}{8}$ d) $\frac{17}{4}$ e) $\frac{47}{6}$

ii) a) $3\frac{1}{8}$ b) $1\frac{5}{6}$ c) $9\frac{1}{3}$ d) $2\frac{5}{7}$ e) $6\frac{4}{5}$

OBJECTIVE I2.2

i) a) $\frac{33}{7} = \frac{28}{7} + \frac{5}{7} = \frac{4}{1} + \frac{5}{7} = 4 + \frac{5}{7} = 4\frac{5}{7}$

b) $\frac{25}{3} = \frac{24}{3} + \frac{1}{3} = \frac{8}{1} + \frac{1}{3} = 8 + \frac{1}{3} = 8\frac{1}{3}$

ii) a) $5\frac{2}{7} = 5 + \frac{2}{7} = \frac{35}{7} + \frac{2}{7} = \frac{37}{7}$

b) $6\frac{3}{7} = 6 + \frac{3}{7} = \frac{42}{7} + \frac{3}{7} = \frac{45}{7}$

OBJECTIVE I3.1

a) $31\frac{13}{60}$ b) $40\frac{1}{30}$ c) $32\frac{41}{144}$ d) $21\frac{4}{5}$ e) $17\frac{13}{36}$

OBJECTIVE I3.2

a) $1\frac{25}{36}$ b) $7\frac{17}{18}$ c) $2\frac{3}{5}$ d) $4\frac{37}{48}$ e) $11\frac{25}{36}$

OBJECTIVE I4.1

a) $n = 1\frac{1}{36}$ b) $p = 2\frac{23}{24}$ c) $m = 21\frac{125}{144}$ d) $a = 22\frac{16}{45}$ e) $z = \frac{11}{12}$

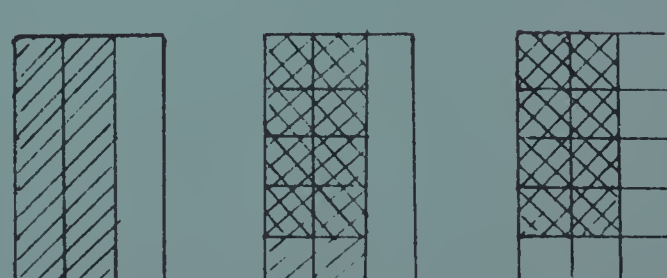
OBJECTIVE I5.1

a) $19\frac{13}{30}$ hr. b) $1\frac{2}{5}$ mi. c) $134\frac{29}{60}$ lb. d) $11\frac{1}{24}$ ft. e) $1\frac{21}{40}$ miles

OBJECTIVE I6.1

a)

b)



OBJECTIVE I7.1

a) 7 b) $\frac{1}{6}$ c) $5\frac{1}{2}$ d) 16 e) $3\frac{1}{7}$

OBJECTIVE I8.1

i) The product of a given number and its reciprocal is one.

ii) a) $\frac{3}{2}$ b) $\frac{1}{4}$ c) 1 d) 2 e) $\frac{2}{7}$

iii) Zero does not have a reciprocal as there is no number which multiplied by 0 gives a product of 1.

OBJECTIVE I9.1

a) $4\frac{2}{3}$ b) 45 c) $3\frac{4}{7}$ d) $\frac{27}{64}$ e) 0 f) 24 g) $\frac{3}{4}$

OBJECTIVE I9.2

$\frac{7}{9} \div \frac{0}{3}$ is not possible as to divide, we replace the division by multiplication by the reciprocal of the divisor, and $\frac{0}{3}$ does NOT have a reciprocal.

OBJECTIVE I10.1

a) m = 22 b) p = $2\frac{1}{4}$ c) q = $2\frac{2}{3}$ d) n = $1\frac{11}{14}$ e) a = $1\frac{35}{36}$

OBJECTIVE I11.1

a) 9 bags b) 260 mi. c) $4\frac{5}{24}$ d) 5 cups

e) $51\frac{9}{20}$ dollars f) $3\frac{3}{5}$

OBJECTIVE I12.1

a) $\frac{1}{2} + (\frac{2}{3} + \frac{1}{2}) = (\frac{1}{2} + \frac{2}{3}) + \frac{1}{2}$

b) $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$

c) $\frac{2}{5} + 0 = \frac{2}{5}$

d) $\frac{3}{4} + \frac{1}{3} = \frac{1}{3} + \frac{3}{4}$

or any similar answers

OBJECTIVE I12.2

i) a) commutative b) associative c) identity d) every non-zero rational number has a reciprocal e) closure f) zero product.

ii) distributive property of multiplication over addition.

OBJECTIVE I12.3

i) every non-zero rational number has a reciprocal.

ii) 5 x ___ = 1 - there is no number to multiply by 5 to get a product of 1.

TOPIC II (OPERATIONS WITH RATIONAL NUMBERS)

PHASE II

ADVANCED LEVEL

Students who achieved nine tenths (90%) or more of the objectives for Phase I are given this material.

First, check your record page. If you missed any of the objectives there, go back through the materials and relearn the proper sections. You will be expected to answer the questions that relate to those objectives on the next test.

In this packet, there are several new objectives that you will be asked to achieve. Since there are only a few of these objectives, they have been collected at the beginning of the section. You should find them more interesting and challenging than the phase I objectives. You will be expected to answer questions on these objectives on the next test.

2A1

OBJECTIVE A1

To state the definition of a non-negative rational number.

Criterion: Statement to include the same ideas as:
"A non-negative rational number is a number named by a fraction of the form $\frac{a}{b}$ where a is a whole number and b is a non-zero whole number."

OBJECTIVE A2

To write the definitions for addition and subtraction of rational numbers named by fractions.

Criterion: Statements including the same ideas as:
"For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} ,"$$

OBJECTIVE A3

To write the definition for multiplication of rational numbers named by fractions.

Criterion: Statement including the same ideas as:
For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} ."$$

OBJECTIVE A4

To write the definition for division of rational numbers named by fractions.

Criterion: Statement including the same ideas as:
"For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ $(\frac{c}{d} \neq 0)$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

OBJECTIVE A5

To write the statements or reasons which complete the proof of a given property of operations for rational numbers.

Example:

For each of the numbered spaces in the proof below write the statement or reason which is missing.

Prove that addition of rational numbers is commutative:

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

Proof:

Statement	Reason
$\frac{a}{b} + \frac{c}{d}$	
$= \underline{\qquad}$ (1)	Definition of addition of rational numbers
$= \frac{da + cb}{db}$	numbers $\underline{\qquad\qquad}$ (2)
$= \frac{\underline{\quad} + \underline{\quad}}{db}$ (3)	Addition of whole numbers is commutative
$= \frac{c}{d} + \frac{a}{b}$	$\underline{\qquad\qquad}$ (4)
$\therefore \frac{a}{b} + \frac{c}{d} = \frac{c}{a} + \frac{a}{b}$	

i.e. Addition of rational numbers is commutative.

Criterion: 75% of computations correct.

SOLUTION

(1) $\frac{ad + bc}{bd}$

(2) Multiplication of rational numbers is commutative.

(3) cb + da

(4) Definition of addition of rational numbers.

2A3

OBJECTIVE A6

To use the distributive property to obtain products of rational numbers.

Example

a) Show how the distributive property may be used to simplify

$$\left(\frac{2}{3} \times \frac{2}{5}\right) + \left(\frac{7}{3} \times \frac{2}{5}\right)$$

b) Show how the distributive property may be used to find the product

$$\frac{1}{2} \times 8\frac{2}{3}$$

c) Find the following product without changing the mixed numerals to fractions:

$$\begin{array}{r} 3\frac{1}{4} \\ \times\ 8\frac{2}{3} \\ \hline \end{array}$$

Criterion: Method correct in each example and no more than one error in computation.

SOLUTION

a) $\left(\frac{2}{3} \times \frac{2}{5}\right) + \left(\frac{7}{3} \times \frac{2}{5}\right) = \left(\frac{2}{3} + \frac{7}{3}\right) \times \frac{2}{5} = \frac{9}{3} \times \frac{2}{5} = 3 \times \frac{2}{5} = \frac{6}{5}$

b) $\frac{1}{2} \times 8\frac{2}{3} = \frac{1}{2} \times \left(8 + \frac{2}{3}\right) = \left(\frac{1}{2} \times 8\right) + \left(\frac{1}{2} \times \frac{2}{3}\right) = 4 + \frac{1}{3} = 4\frac{1}{3}$

c)
$$\begin{array}{r} 3\frac{1}{6} \\ \times\ 4\frac{2}{3} \\ \hline 2\frac{2}{18} \\ 12\ \frac{4}{6} \\ \hline 14\frac{14}{18} \end{array} = 14\frac{7}{9}$$

1. DEFINITION FOR RATIONAL NUMBERS

In mathematics, a definition is a statement which tells precisely what something means.

In this section we will see the definition for rational numbers.

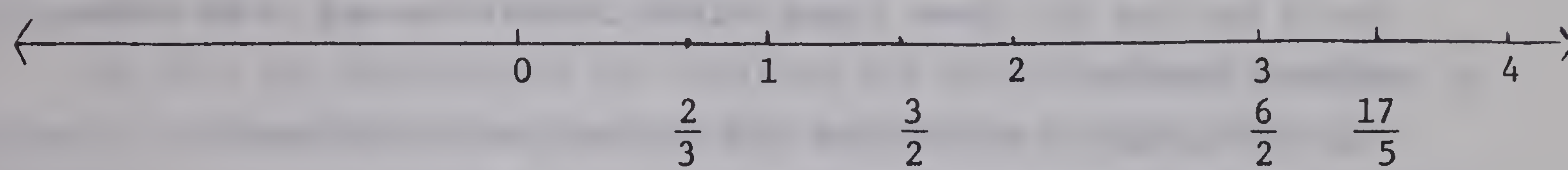
You have seen that each rational number is associated with an infinite set of equivalent fractions each of which has the form $\frac{a}{b}$ where a is a whole number and b is a non-zero whole number (or natural number).

e.g. The rational number two thirds is associated with the infinite set of fractions $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\}$

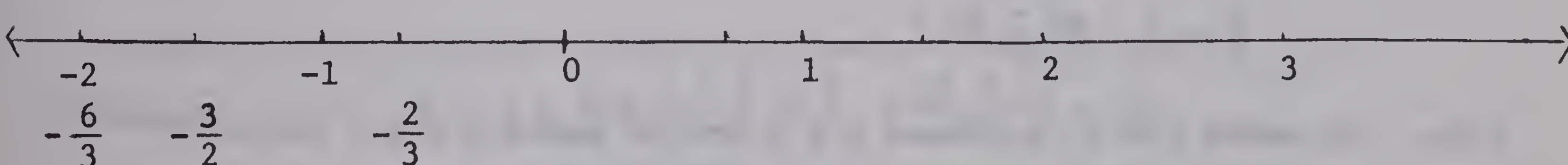
This gives us a way of defining rational numbers as follows:

A rational number is a number named by a fraction of the form $\frac{a}{b}$ where a is a whole number and b is a non-zero whole number.

This defines the sort of rational number which we have been using. That is, 0 and the rational numbers associated with points to the right of the point associated with 0 on the number line.



There are however rational numbers associated with points to the left of the point associated with 0 on the number line. Some of these are shown below.



Note the symbols used for these new rational numbers.

The two types of rational numbers are distinguished by calling those associated with points to the right of the point associated with 0 the POSITIVE RATIONAL NUMBERS and those associated with points to the left of the point associated with 0 the NEGATIVE RATIONAL NUMBERS.

You will learn more about the negative rational numbers later.

0 is neither a positive rational number nor a negative rational number.

0 and the positive rational numbers make up a set called the set of non-negative rational numbers (i.e. the rational numbers which are not negative).

The set of rational numbers we defined earlier is actually this set of non-negative rational numbers.

2. DEFINITIONS FOR ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

In this section we will give definitions of addition and subtraction of rational numbers named by fractions; i.e. mathematical statements which give precisely the meaning of addition and subtraction of rational numbers named by fractions.

Apart from stating precisely what something means, a definition is a general statement which represents all possible particular instances of the thing being defined.

Let's see how the above ideas relate to addition and subtraction of rational numbers.

The definitions for addition and subtraction of rational numbers named by fractions are:

"For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}\text{"}$$

Note: ad means a x d, bc means b x c and bd means b x d. Pairs of letters written together as in the definitions indicate multiplication.

These may not look quite like the addition or subtraction we did in SECTION 1, however, they are general statements and do give the result for any addition or subtraction example.

1. Check that the sum of $\frac{5}{6}$ and $\frac{2}{5}$ is the same by the definition and by the common denominator method.
2. Check that the difference of $\frac{5}{6}$ and $\frac{3}{4}$ is the same by the definition and by the common denominator method.

Using the definition, you may not get the result as a basic fraction. However, a fraction and its equivalent basic fraction do name the same rational number.

Either the definition, or the common denominator method can be used to find the sum or difference of two rational numbers. We usually use the common denominator method (least common denominator in fact). However this method cannot be readily stated in the form of a general definition.

3. DEFINITION OF MULTIPLICATION OF RATIONAL NUMBERS

The definition for multiplication of rational numbers named by fractions is as given in OBJECTIVE A7.1.

As with the definitions for addition and subtraction of rational numbers named by fractions, the product may not be a basic fraction. However, we saw in SECTION 7 how the product can be reduced to a basic fraction.

Answers:

1. $$\frac{5}{6} + \frac{2}{5} = \frac{5 \times 5 + 6 \times 2}{6 \times 5} = \frac{25 + 12}{30} = \frac{37}{30}$$

$$\frac{5}{6} + \frac{2}{5} = \frac{25}{30} + \frac{12}{30} = \frac{37}{30}$$

2. $$\frac{5}{6} - \frac{3}{4} = \frac{5 \times 4 - 6 \times 3}{6 \times 4} = \frac{20 - 18}{24} = \frac{2}{24} = \frac{1}{12}$$

$$\frac{5}{6} - \frac{3}{4} = \frac{10}{12} - \frac{9}{12} = \frac{1}{12}$$

4. DEFINITION OF DIVISION OF RATIONAL NUMBERS

The definition for division of rational numbers named by fractions is as given in OBJECTIVE A9.1. Note the condition that the divisor cannot be zero since we cannot divide by 0.

5. PROVING PROPERTIES OF OPERATIONS WITH RATIONAL NUMBERS

In SECTION 12, we used examples to check that various properties of addition and multiplication of rational numbers named by fractions held.

We can in fact prove that these properties hold by using

(1) our knowledge of the properties of addition and multiplication of whole numbers (stated in SECTION 12)

and (2) the definitions for addition and multiplication of rational numbers named by fractions (stated in OBJECTIVES A1.1 and A7.1).

Let's see how two of the properties can be proved. Then you can complete the proofs of the others.

I Prove that addition of rational numbers is associative.

Let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ be any rational numbers.

Prove: $(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$

Proof:

Statement	Reason	Comment
$(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{(ad + bc)}{bd} + \frac{e}{f}$	definition of addition	
$= \frac{(ad + bc)f + (bd)e}{(bd)f}$	definition of addition	(bd)e means (b x d)xe
$= \frac{f(ad + bc) + (bd)e}{(bd)f}$	Mult. of whole numbers is commutative	Change of order.
$= \frac{\{f(ad) + f(bc)\} + (bd)e}{(bd)f}$	distributive property for whole numbers	
$= \frac{\{(ad)f + (bc)f\} + (bd)e}{(bd)f}$	Mult. of whole numbers is commutative	Change of order.
$= \frac{(ad)f + (bc)f + (bd)e}{(bd)f}$	Add. of whole numbers is associative	Grouping for addition does not matter.
$= \frac{adf + bcf + bde}{bdf}$	Mult. of whole numbers is associative.	Grouping for multiplication does not matter.

$\frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$	
$= \frac{a}{b} + \frac{(cf + de)}{df}$	definition of addition
$= \frac{a(df) + b(cf + de)}{b(df)}$	definition of addition
$= \frac{a(df) + \{b(cf) + b(de)\}}{b(df)}$	distributive property for whole numbers.
$= \frac{a(df) + b(cf) + bde}{b(df)}$	Add. of whole numbers is associative.
$= \frac{adf + bcf + bde}{bdf}$	Mult. of whole numbers is associative.
$\therefore\ (\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f})$	

The last line in each set of equalities is the same.

$\therefore$ Addition of rational numbers is associative.

$\therefore$ is an abbreviation for 'therefore'. However, it is never used in an English sentence.

A proof, as you see above, is a sequence of statements for each of which a reason is given. Proofs can often be conveniently given in two column form. The first column gives the statements; the second column gives the reasons for the statement.

In our proof above, the reasons used are the definition of addition for rational numbers and the properties of operations for whole numbers.

II Prove that 1 is the identity for multiplication of rational numbers.

Let $\frac{a}{b}$ be any rational number.

Prove: $\frac{a}{b} \times 1 = \frac{a}{b}$

Proof:

Statement	Reason
$\frac{a}{b} \times 1$	$\frac{1}{1}$ is another name for 1.
$= \frac{a}{b} \times \frac{1}{1}$	definition of multiplication
$= \frac{a \times 1}{b \times 1}$	
$= \frac{a}{b}$	1 is the identity for multiplication of whole numbers.

$\therefore$ 1 is the identity for multiplication of rational numbers.

Now turn to and do the activities on the following pages.

Each of the following is a proof for one of the properties of operations with rational numbers. In these proofs, certain statements or reasons have been omitted and left for you to give. Each place where something has been omitted is marked with an * and numbered. Write your answers in your workbook, NOT on these pages. At the end of each proof, check the answers you gave with the answers given at the end of the advanced work. If you cannot see why a particular answer is given, ask another person in the A group or ask your teacher.

The symbol "$\therefore$" means "therefore".

1. Prove that the rational numbers are closed for addition.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} + \frac{c}{d}$ is a rational number.

Proof:

Statement	Reason	Comment
$\frac{a}{b} + \frac{c}{d}$ $= \frac{ad + bc}{bd}$	Definition of addition of rational numbers	
ad and bc are whole numbers	Whole numbers are closed for multiplication	Here we make use of properties of whole numbers.
(ad + bc) is a whole number	____________ *1	
bd is a non-zero whole number	$b \neq 0$, $d \neq 0$ and _____ *2	
$\therefore \frac{ad + bc}{bd}$ is a rational number	Definition of rational number.	
i.e. $\frac{a}{b} + \frac{c}{d}$ is a rational number		This comes from the first two lines in the proof.
i.e. The rational numbers are closed for addition.		

2. Prove that the rational numbers are closed for multiplication.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} \times \frac{c}{d}$ is a rational number.

Proof:

Statement	Reason	Comment
$\frac{a}{b} \times \frac{c}{d}$		
$= \frac{ac}{bd}$	______________ *3	ac means a x c
ac is a whole number	______________ *4	
______________ *5	b≠0, d≠0, and whole numbers are closed for multiplication.	The denominator must be non-zero.
$\therefore \frac{ac}{bd}$ is a rational number.	______________ *6	
i.e. $\frac{a}{b} \times \frac{c}{d}$ is a rational number.		This comes from the first two lines of the proof.
i.e. The rational numbers are closed for multiplication.		

3. Prove that multiplication of rational numbers is commutative.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$

Proof:

Statement	Reason	
$\frac{a}{b} \times \frac{c}{d}$		
$= \frac{ac}{bd}$	Definition of multiplication of rational numbers	
$= \frac{ca}{db}$	______________ *7	The order is changed.
= ______________ *8	Definition of multiplication of rational numbers.	Here the definition of multiplication of rational numbers is used in the reverse direction to its use in the first two lines of the proof.
$\therefore \frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$		
i.e. ______________ *9		

4. Prove that addition of rational numbers is commutative.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers.

Prove: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

Proof:

Statement	Reason	Comment
$\frac{a}{b} + \frac{c}{d}$		
$=$ ____________________ *10	Definition of addition of rational numbers.	
$= \frac{da + cb}{db}$	____________________ *11	This property is used three times here.
$=$ ____________________ *12	Addition of whole numbers is cummutative.	
$= \frac{c}{d} + \frac{a}{b}$	____________________ *13	
$\therefore$ __________ = _____ *14		
i.e. Addition of rational numbers is commutative		

5. Prove that multiplication of rational numbers is associative.

Let $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ be any rational numbers.

Prove: $(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$

Proof:

Statement	Reason	Comment
$(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f}$		
$= (\frac{ac}{bd}) \times \frac{e}{f}$	____________________ *15	The definition is used twice here.
$= \frac{(ac)\ e}{(bd)\ f}$	Definition of multiplication of rational numbers.	
$= \frac{(\quad)}{(\quad)}$ *16	Mult. of whole numbers is associative.	The grouping is changed.
$= \frac{a}{b} \times (\frac{cd}{df})$	____________________ *17	
$=$ ______________ *18	Definition of mult. of rational numbers	
$\therefore (\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$		
i.e. __________________________ *19		

2A12

Note: In proof I on page of the description we proved that addition of rational numbers is associative.

6. Prove that 0 is the identity for addition of rational numbers.

Let $\frac{a}{b}$ be any rational number.

Prove: $\frac{a}{b} + 0 = \frac{a}{b}$

Proof:

Statement	Reason	Comment
$\frac{a}{b} + 0$		
$= \frac{a}{b} + \frac{0}{1}$		$\frac{0}{1}$ is another name for 0.
$= \frac{a \times 1 + b \times 0}{b \times 1}$	______ *20	
$= \frac{\quad + \quad}{\quad}$ *21	1 is the identity for mult. of whole numbers and $b \times 0 = 0$	
$= \frac{a}{b}$	______ *22	Another property of whole numbers.
$\therefore$ ______ *23		
i.e. 0 is the identity for addition of rational numbers.		

7. Prove that multiplication is distributive over addition of rational numbers.

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers.

Prove: $\frac{a}{b} \times (\frac{c}{d} + \frac{e}{f}) = (\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{f})$

Proof:

Statement	Reason	Comment
$\frac{a}{b} \times (\frac{c}{d} + \frac{e}{f})$		
$= \frac{a}{b} \times (\frac{\quad}{\quad})$ *24	Definition of addition of rational numbers.	
$= \frac{a\ (cf + de)}{b(df)}$	______ *25	
$= \frac{a(cf) + a(de)}{b(df)}$	______ *26	
$= \frac{acf + ade}{bdf}$	Multiplication of whole numbers is associative.	This permits the parentheses to be omitt

Statement	Reason	Comment
$(\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{f})$		Now we start with the right side of what we have to prove and get it into the same form as that to which we changed the left side.
$= \frac{ac}{bd} + \frac{ae}{bf}$	____________ *27	
$= \frac{(ac)(bf) + (bd)(ae)}{(bd)(bf)}$	____________ *28	
$= \frac{acbf + bdae}{bdbf}$	____________ *29	
$= \frac{bacf + bdae}{bdbf}$	Mult. of whole numbers is commutative.	The order in the products in the numerator has been changed.
$= \frac{b(acf + dae)}{bdbf}$	Distributive property for whole numbers.	
$= \frac{b(acf + dae)}{bdbf}$	Numerator and denominator are divided by b.	The fraction is reduced.
$= \frac{acf + dae}{dbf}$		
$= \frac{acf + ade}{bdf}$	____________ *30	The order is changed in two of the products This is now the same as the left side.

$\therefore \frac{a}{b} \times (\frac{c}{d} + \frac{e}{f}) =$ ____________ *31

i.e. Multiplication is distributive over addition of rational numbers.

8. Prove that the multiplication property of zero holds for the rational numbers.

Let $\frac{a}{b}$ be any rational number.

Prove: $\frac{a}{b} \times 0 = 0$

Proof:

Statement	Reason	Comment
$\frac{a}{b} \times 0$		
$= \frac{a}{b} \times \frac{0}{1}$		$\frac{0}{1}$ is another name for 0
$=$ ________ *32	Definition of mult. of rational numbers.	
$= \frac{0}{b}$	____________ and ____________ *33	$\frac{0}{b}$ is another name for 0.
$= 0$		
$\therefore \frac{a}{b} \times 0 = 0$		
i.e. ________________ *34		

6. USE OF THE DISTRIBUTIVE PROPERTY

Multiplication and addition of whole numbers and of rational numbers are related by the distributive property.

We have seen the distributive property in the form:

$a \times (b + c) = (a \times b) + (a \times c)$

Now look at the following:

$$\begin{aligned} (b + c) \times a &= a \times (b + c) && \text{Commutative property of multiplication.} \\ &= (a \times b) + (a \times c) && \text{Distributive property} \\ &= (b \times a) + (c \times a) && \text{Commutative property of multiplication.} \end{aligned}$$

i.e. $(b + c) \times a = (b \times a) + (c \times a)$

We see that the distributive property can be used when the single multiplier is on the left or right of the parentheses.

Now let us look at some examples in which the distributive property is used.

Examples

(1) $6 \times 37 = 6 \times (30 + 7) = (6 \times 30) + (6 \times 7) = 180 + 42 = 222$

(2) $6 \times 3\frac{1}{4} = 6 \times (3 + \frac{1}{4}) = (6 \times 3) + (6 \times \frac{1}{4}) = 18 + \frac{6}{4} = 18 + \frac{3}{2} = 19\frac{1}{2}$

(3) $\frac{1}{3} \times 6\frac{3}{5} = \frac{1}{3} \times (6 + \frac{3}{5}) = (\frac{1}{3} \times 6) + (\frac{1}{3} \times \frac{3}{5}) = 2 + \frac{1}{5} = 2\frac{1}{5}$

(4) $2\frac{1}{4} \times 5 = (2 + \frac{1}{4}) \times 5 = (2 \times 5) + (\frac{1}{4} \times 5) = 10 + \frac{5}{4} = 10 + 1\frac{1}{4} = 11\frac{1}{4}$

1. The product $3\frac{1}{3} \times 6\frac{1}{2}$ can be written as

$$\begin{array}{r} 6\frac{1}{2} \\ \times\ 3\frac{1}{3} \\ \hline \end{array}$$

Examine the following to see what has been done to find the product.

$$\begin{array}{r} 6\frac{1}{4} \\ \times\ 3\frac{1}{3} \\ \hline 2\frac{1}{12} \\ 18\frac{3}{4} \\ \hline 20\frac{10}{12} \end{array} = 20\frac{5}{6}$$

2. Use the same method as above to show that

$4\frac{1}{3} \times 2\frac{1}{2} = 10\frac{5}{6}$.

In the examples in items 1 and 2 on the previous page, we have also been using the distrubutive property. The following shows how.

$$3\frac{1}{3} \times 6\frac{1}{4} = (3 + \frac{1}{3}) \times 6\frac{1}{4} = (3 \times 6\frac{1}{4}) + (\frac{1}{3} \times 6\frac{1}{4}) = 3 \times (6 + \frac{1}{4}) + \frac{1}{3} \times (6 + \frac{1}{4})$$

$$= (3 \times 6) + (3 \times \frac{1}{4}) + (\frac{1}{3} \times 6) + (\frac{1}{3} \times \frac{1}{4}) = 18 + \frac{3}{4} + 2 + \frac{1}{12}$$

$$= 20 + \frac{9}{12} + \frac{1}{12} = 20 + \frac{10}{12} = 20\frac{5}{6}$$

Now do the exercises on the following activity page.

Answers: 1. $\frac{1}{3} \times 6\frac{1}{4} = \frac{1}{3} \times (6 + \frac{1}{4}) = (\frac{1}{3} \times 6) + (\frac{1}{3} \times \frac{1}{4}) = 2 + \frac{1}{12} = 2\frac{1}{12}$

$3 \times 6\frac{1}{4} = 3 \times (6 + \frac{1}{4}) = (3 \times 6) + (3 \times \frac{1}{4}) = 18 + \frac{3}{4} = 18\frac{3}{4}$

$2\frac{1}{12} + 18\frac{3}{4} = 20 + \frac{1}{12} + \frac{9}{12} = 20 + \frac{10}{12} = 20\frac{5}{6}$

2.
$$\begin{array}{r} 4\frac{1}{3} \\ \times 2\frac{1}{2} \\ \hline 2\frac{1}{6} \\ 8\frac{2}{3} \\ \hline 10\frac{5}{6} \end{array}$$

SECTION 6 - ACTIVITY

1. Use the distributive property to simplify the following:

(a) $(\frac{3}{4} \times \frac{2}{5}) + (\frac{3}{4} \times \frac{3}{5})$

(b) $(\frac{5}{8} \times \frac{5}{6}) + (\frac{5}{8} \times \frac{11}{6})$

(c) $(\frac{2}{3} \times \frac{3}{8}) + (\frac{2}{3} \times 2\frac{5}{8})$

(d) $(1\frac{3}{5} \times \frac{3}{4}) + (1\frac{3}{5} \times \frac{1}{4})$

(e) $(2\frac{1}{3} \times 1\frac{3}{5}) + (2\frac{1}{3} \times 1\frac{2}{5})$

(f) $(\frac{7}{8} \times \frac{1}{3}) + (\frac{1}{8} \times \frac{1}{3})$

(g) $(\frac{3}{10} \times \frac{1}{2}) + (\frac{17}{10} \times \frac{1}{2})$

(h) $(4\frac{1}{4} \times \frac{3}{8}) + (3\frac{3}{4} \times \frac{3}{8})$

(i) $(1\frac{2}{5} \times 1\frac{1}{3}) + (1\frac{3}{5} \times 1\frac{1}{3})$

2. Use the distributive property to find the products.

(a) $5 \times 6\frac{1}{5}$ (d) $\frac{5}{8} \times 16\frac{2}{5}$ (g) $6\frac{1}{4} \times \frac{1}{2}$

(b) $8 \times 2\frac{3}{4}$ (e) $2\frac{1}{3} \times 6$ (h) $9\frac{3}{8} \times \frac{2}{3}$

(c) $\frac{3}{4} \times 12\frac{1}{3}$ (f) $3\frac{2}{5} \times 10$

3. Find the following products without changing mixed numerals to fractions.

(a) $4\frac{1}{2} \times 1\frac{1}{4}$

(b) $12\frac{1}{3} \times 2\frac{3}{4}$

(c) $4\frac{1}{9} \times 6\frac{3}{4}$

(d) $10\frac{1}{4} \times 3\frac{4}{5}$

(e) $8\frac{1}{5} \times 2\frac{5}{8}$

(f) $15\frac{1}{4} \times 4\frac{3}{5}$

(g) $6\frac{3}{5} \times 2\frac{2}{3}$

SOLUTIONS

SECTION 5 - ACTIVITY

1. Whole numbers are closed for addition (ad and bc are both whole numbers from the previous line.)
2. Whole numbers are closed for multiplication.
3. Definition of multiplication of rational numbers.
4. Whole numbers are closed for multiplication.
5. bd is a non-zero whole number.
6. Definition of a rational number.
7. Multiplication of whole numbers is commutative.
8. $\frac{c}{d} \times \frac{a}{b}$
9. Multiplication of rational numbers is commutative.
10. $\frac{ad + bc}{bd}$
11. Multiplication of whole numbers is commutative.
12. $\frac{cb + da}{db}$
13. Definition of addition of rational numbers.
14. $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$
15. Definition of multiplication of rational numbers.
16. $\frac{a(ce)}{b(df)}$
17. Definition of multiplication of rational numbers
18. $\frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$
19. Multiplication of rational numbers is associative.
20. Definition of addition of rational numbers.
21. $\frac{a + 0}{b}$
22. 0 is the identity for addition of whole numbers.
23. $\frac{a}{b} + 0 = \frac{a}{b}$
24. $\frac{cf + de}{df}$
25. Definition of multiplication of rational numbers.
26. Distributive property for whole numbers.
27. Definition of multiplication of rational numbers.
28. Definition of addition of rational numbers.
29. Multiplication of whole numbers is associative.
30. Multiplication of whole numbers is commutative.
31. $(\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{f})$
32. $\frac{a \times 0}{b \times 1}$
33. Multiplication property of 0 for whole numbers and 1 is the identity for multiplication of whole numbers.
34. The multiplication property of 0 holds for the rational numbers.

2A18

ANSWERS

SECTION 6 - ACTIVITY

1. (a) $\frac{3}{4} \times (\frac{2}{5} + \frac{3}{5}) = \frac{3}{4} \times 1 = \frac{3}{4}$
   (b) $\frac{5}{8} \times (\frac{5}{6} + \frac{11}{6}) = \frac{5}{8} \times 2 = \frac{5}{4}$
   (c) $\frac{2}{3} \times (\frac{3}{8} + 2\frac{5}{8}) = \frac{2}{3} \times 3 = 2$
   (d) $1\frac{3}{5} \times (\frac{3}{4} + \frac{1}{4}) = 1\frac{3}{5} \times 1 = 1\frac{3}{5}$
   (e) $2\frac{1}{3} \times (1\frac{3}{5} + 1\frac{2}{5}) = 2\frac{1}{3} \times 3 = 7$
   (f) $(\frac{7}{8} + \frac{1}{8}) \times \frac{1}{3} = 1 \times \frac{1}{3} = \frac{1}{3}$
   (g) $(\frac{3}{10} + \frac{17}{10}) \times \frac{1}{2} = 2 \times \frac{1}{2} = 1$
   (h) $(4\frac{1}{4} + 3\frac{3}{4}) \times \frac{3}{8} = 8 \times \frac{3}{8} = 3$
   (i) $(1\frac{2}{5} + 1\frac{3}{5}) \times 1\frac{1}{3} = 3 \times 1\frac{1}{3} = 4$

2. (a) $5 \times (6 + \frac{1}{5}) = (5 \times 6) + (5 \times \frac{1}{5}) = 30 + 1 = 31$
   (b) $8 \times (2 + \frac{3}{4}) = (8 \times 2) + (8 \times \frac{3}{4}) = 16 + 6 = 22$
   (c) $\frac{3}{4} \times (12 + \frac{1}{3}) = (\frac{3}{4} \times 12) + (\frac{3}{4} \times \frac{1}{3}) = 9 + \frac{1}{4} = 9\frac{1}{4}$
   (d) $\frac{5}{8} \times (16 + \frac{2}{5}) = (\frac{5}{8} \times 16) + (\frac{5}{8} \times \frac{2}{5}) = 10 + \frac{1}{4} = 10\frac{1}{4}$
   (e) $(2 + \frac{1}{3}) \times 6 = (2 \times 6) + (\frac{1}{3} \times 6) = 12 + 2 = 14$
   (f) $(3 + \frac{2}{5}) \times 10 = (3 \times 10) + (\frac{2}{5} \times 10) = 30 + 4 = 34$
   (g) $(6 + \frac{1}{4}) \times \frac{1}{2} = (6 \times \frac{1}{2}) + (\frac{1}{4} \times \frac{1}{2}) = 3 + \frac{1}{8} = 3\frac{1}{8}$
   (h) $(9 + \frac{3}{8}) \times \frac{2}{3} = (9 \times \frac{2}{3}) + (\frac{3}{8} \times \frac{2}{3}) = 6 + \frac{1}{4} = 6\frac{1}{4}$

3. (a)
$$\begin{array}{r} 4\frac{1}{2} \\ \times\ 1\frac{1}{4} \\ \hline 1\frac{1}{8} \\ 4\frac{1}{2} \\ \hline 5\frac{5}{8} \end{array}$$

(b)
$$\begin{array}{r} 12\frac{1}{3} \\ \times\ 2\frac{3}{4} \\ \hline 9\frac{1}{4} \\ 24\frac{2}{3} \\ \hline 33\frac{11}{12} \end{array}$$

(c)
$$\begin{array}{r} 4\frac{1}{9} \\ \times\ 6\frac{3}{4} \\ \hline 3\frac{1}{12} \\ 24\ \frac{2}{3} \\ \hline 27\frac{9}{12} \end{array} = 27\frac{3}{4}$$

(d)
$$\begin{array}{r} 10\frac{1}{4} \\ \times\ 3\frac{4}{5} \\ \hline 8\frac{1}{5} \\ 30\frac{3}{4} \\ \hline 38\frac{19}{20} \end{array}$$

(e)
$$\begin{array}{r} 8\frac{1}{5} \\ \times\ 2\frac{5}{8} \\ \hline 5\frac{1}{8} \\ 16\frac{2}{5} \\ \hline 21\frac{21}{40} \end{array}$$

(f)
$$\begin{array}{r} 15\frac{1}{4} \\ \times\ 4\frac{3}{5} \\ \hline 9\frac{3}{20} \\ 61 \\ \hline 70\frac{3}{20} \end{array}$$

(g)
$$\begin{array}{r} 6\frac{3}{5} \\ \times\ 2\frac{2}{3} \\ \hline 4\frac{2}{5} \\ 13\frac{1}{5} \\ \hline 17\frac{3}{5} \end{array}$$

Topic II

Post-Test II - Basic (Form B)

1. Show all your work in the spaces provided.
2. Place your answers or solutions on the lines (_______) whenever they are provided for this purpose on the right hand side of the page.
3. Your items will be marked wrong if your solution does not appear in the space indicated.
4. Work carefully and do not spend too much time on any one item. All items should be attempted.

B1.1 Find the sums of the following and write each as a basic fraction.

A. $$\begin{array}{r} \frac{5}{7} \\ \frac{2}{3} \\ + \frac{4}{21} \\ \hline \end{array}$$

B. $\frac{4}{9} + \frac{5}{6} + \frac{2}{3}$

B1.1 A. (_______)

B. (_______)

B1.2 Find the differences of the following and write each as a basic fraction.

A. $$\begin{array}{r} \frac{14}{12} \\ - \frac{1}{3} \\ \hline \end{array}$$

B. $\frac{7}{3} - \frac{4}{5}$

B1.2 A. (_______)

B. (_______)

B2.1 A. Write the fraction $\frac{29}{4}$ as a mixed numeral.

B2.1 A. (_______)

B. Write $4\frac{5}{7}$ as a fraction.

B2.1 B. (_______)

B 3.1 Find the sums of the following and write the fraction part in the answer as a basic fraction.

A. $9\frac{8}{9}$ B. $9\frac{5}{12} + 4\frac{5}{8} + \frac{2}{3}$

$18\frac{7}{27}$

______

B3.1 A(__________)

B3.1 B(__________)

B 3.2 Find the difference of the following and write the fraction part as a basic fraction.

A. $8\frac{1}{5} - 7\frac{3}{4}$ B. $30\frac{4}{15}$

$- 12\frac{1}{6}$

______

B3.2 A(__________)

B3.2 B(__________)

B 4.2 Solve the condition $1\frac{3}{4} = m + \frac{7}{6}$ and write the fraction in the solution as a basic fraction. Show your check.

B4.1 (__________)

B 5.1 Jim is 5 feet $3\frac{1}{2}$ inches tall. Albert is 5 feet $7\frac{1}{5}$ inches tall. How many inches is Albert taller than Jim?

B5.1 (__________)

B 7.1 Find the basic fraction for each product

A. $\frac{13}{27} \times 18 \times \frac{33}{26}$

B. $\frac{7}{16}$ of $3\frac{3}{7}$

B7.1 A(__________)

B7.1 B(__________)

B 8.1 Give the reciprocal for each of the following numbers if it exists.

A. $\frac{3}{13}$ B. $\frac{0}{27}$

C. 25 D. $\frac{19}{17}$

B8.1 A(__________)

B8.1 B(__________)

B8.1 C(__________)

B8.1 D(__________)

B 9.1 Find the quotients and give them as basic fractions.

A. $\frac{24}{25} \div \frac{27}{20}$

B. $\dfrac{\frac{15}{8}}{\frac{25}{28}}$

B9.1 A(__________)

B9.1 B(__________)

B 10.1 Solve the condition $\frac{3}{5}a = \frac{27}{35}$ and write the solution as a basic fraction. Give your check.

B10.1 (__________)

B 11.1 Barry and Brent weighed themselves. They posed their weights as a puzzler to their class. Barry claimed that he weighed $\frac{7}{8}$ of 64 lbs. and Brent said his weight was the same as 72 lbs. divided by $\frac{9}{7}$. Their question was: "Which one of us is the heaviest?"

B11.1 (__________)

Topic II

Post-Test II - Intermediate (Form B)

1. Show all your work in the spaced provided.
2. Place your answers or solutions on the lines (_____) whenever they are provided for this purpose on the right hand side of the page.
3. Your items will be marked wrong if your solution does not appear in the space indicated.
4. Work carefully and do not spend too much time on any one question. All items should be attempted.

I1.1 Find the sums of the following and write each as a basic fraction.

A. $\frac{7}{5} + \frac{8}{3} + \frac{14}{9}$

B.
$$\begin{array}{r} \frac{15}{7} \\ \frac{9}{5} \\ \frac{3}{35} \\ \hline \end{array}$$

I1.1 A(__________)

I1.1 B(__________)

I1.2 Find the difference of the following and write each as a basic fraction.

A. $\frac{11}{12} - \frac{4}{5}$

B.
$$\begin{array}{r} \frac{25}{24} \\ - \frac{13}{16} \\ \hline \end{array}$$

I1.2 A(__________)

I1.2 B(__________)

I2.2 Justify that the mixed numeral for $\frac{63}{12}$ is $5\frac{1}{4}$ by using fractions.

B. Using fractions justify that the fraction for $5\frac{1}{3}$ is $\frac{16}{3}$.

I 3.2 Find the difference for the following and write the fraction part in each answer as a basic fraction.

A. $45\frac{11}{16} - 15\frac{7}{10}$

B. $17\frac{7}{8}$

$- 3\frac{7}{20}$

I3.2 A(__________)

I3.2 B(__________)

I 4.1 Solve the condition $9\frac{9}{16} = m + 3\frac{19}{24}$. Write the fraction in the solution as a basic fraction and show your check.

I4.1 (__________)

I 5.1 Brent is repainting his bike. He uses a mixture of two colours. He has mixed $\frac{3}{16}$ pint red paint and $\frac{1}{3}$ pint blue paint. he finds that he needs $\frac{11}{12}$ pint of the mixture. How much is he short?

I5.1 (__________)

I 6.1 Using diagrams show how $\frac{2}{3}$ of $\frac{3}{5}$ can be obtained.

I 7.1 Find the products for each of the following and write the fractions in the answers as basic fractions.

A. $\frac{7}{18} \times 1\frac{23}{58} \times 29$ B. $\frac{4}{21}$ of $18\frac{3}{4}$

I7.1 A(__________)

I7.1 B(__________)

I 8.1 In the following illustrate with an example:

A. (1) Write a whole number other than zero and also write its reciprocal.

(2) Write a rational number which is not a whole number and also write its reciprocal.

B. State the property which a number and its reciprocal have.

C. Explain why zero has no reciprocal.

I 9.1 Find the quotient of the following:

A. $5\frac{3}{5} - 1\frac{7}{20}$ B. $\dfrac{5\frac{1}{3}}{1\frac{1}{9}}$

I9.1 A(________)

I9.1 B(________)

I 9.2 Using reciprocals explain why you cannot divide by zero.

I 10.1 Solve the condition $\frac{63}{16} = \frac{35}{32}n$. Write the fraction part in the solution as a basic fraction. Show the check.

I10.1 (________)

I11.1 Twigge went to the store to buy some diet foods. The store had a sale on. Brand A was on sale for 3 tins for 99 cents and Brand B was on sale for three tins for 90 cents. Brand A has $4\frac{1}{4}$ oz. tins. Brand B has $3\frac{3}{4}$ oz. tins. Which brand is the best buy? Show all work.

I11.1 (________)

Topic II

Intermediate - Supplement

Post - test II

I 12.1 Match the example given in column 1 with the correct property from column 2 for the properties of addition of rational numbers.

____	$\frac{1}{2} + \frac{2}{3}$ R	a. Associative property
____	$\frac{1}{2} + 0 = \frac{1}{2}$	b. Commutative property
____	$\frac{2}{3} + \frac{3}{4} = \frac{3}{4} + \frac{2}{3}$	c. Closure property
____	$(\frac{1}{2} + \frac{1}{3}) + \frac{1}{4} = \frac{1}{2} + (\frac{1}{3} + \frac{1}{4})$	d. Identity property

I 12.2 Match the example given in column 1 with the correct property from column 2 for the properties of multiplication of rational numbers.

____	$\frac{1}{2} \times \frac{2}{3} = \frac{2}{3} \times \frac{1}{2}$	a. Closure property
____	$\frac{1}{2}(\frac{1}{4} + \frac{1}{5}) = \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{5}$	b. Associative property
____	$\frac{2}{3} \times \frac{3}{2} = 1$	c. Identity property
____	$1 \times \frac{3}{4} = \frac{3}{4}$	d. Commutative property
____	$\frac{1}{2} \times \frac{3}{4}$ R	e. Distributive property
____	$(\frac{1}{2} \times \frac{1}{3}) \times \frac{1}{4} = \frac{1}{2} \times (\frac{1}{3} \times \frac{1}{4})$	f. Reciprocal property

I 12.3 State the property of multiplication the rational numbers have that the whole numbers do not have. Give an example to illustrate your statement.

Topic II

Post-Test II - Advanced (Form B)

Show your work in the spaces provided for this purpose. Work carefully and do not spend too much time on any one question.

A1 Define a non-negative rational number.

A2 Write the definitions for addition and subtraction of rational numbers named by fractions.

A3 Write the definition formultiplication of rational numbers named by fractions.

A4 Write the definition for division of rational numbers named by fractions.

A5 Complete the following proof

Prove that addition is commutative for rational numbers.

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two rational numbers

Prove: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$

Proof:

Statement	Reason
$\frac{a}{b} + \frac{c}{d}$	
= (1) ________________.	Definition for addition of rational numbers.
= (2) ________________.	Multiplication of whole numbers is commutative.
= $\frac{cb + da}{db}$	(3) ________________
= $\frac{c}{d} + \frac{a}{b}$	(4) ________________

$\therefore$ (5)________ = ________

ie. Addition for rational numbers is commutative.

A6 A. Show how the distributive property may be used to find the product

$$\frac{8}{15} \times 30\frac{45}{88}$$

B. Find the product without changing the mixed numerals to fractions.

$$\begin{array}{r} 9\frac{3}{8} \\ \times\ 4\frac{2}{3} \\ \hline \end{array}$$

C. Show how the distributive property may be used to simplify

$$\left(\frac{9}{46} \times \frac{44}{63}\right) + \left(\frac{9}{46} \times \frac{16}{21}\right)$$

Think back to Problem Solving in Topic 1. It was stated that there are no 'hard-and-fast' rules to follow in the solving of all problems. Each problem will be different and will require its own 'line-of-attack'.

Sometimes a mathematical pattern or relationship can be discovered if we first look at several numerical examples. Here is an example in which looking at examples helps.

The first of two rational numbers is greater than zero, and the product of the two rational numbers is greater than zero.

(I) If the product is less than the first number, what must be true about the second number?

(II) If the product is greater than the first number, what must be true about the second number?

Let's look at some numerical examples to determine what must be true about the second number in each case.

(I)

First Number	Second Number	Product (less than first number)	Finding the Second Number	Second Number
5	n	3	$5 \times n = 3$ $\frac{1}{5} \times 5 \times n = \frac{1}{5} \times 3$ $n = \frac{3}{5}$	$\frac{3}{5}$
7	p	4	$7 \times p = 4$ $\frac{1}{7} \times 7 \times p = \frac{1}{7} \times 4$ $p = \frac{4}{7}$	$\frac{4}{7}$
$\frac{2}{3}$	q	$\frac{1}{4}$	$\frac{2}{3} \times q = \frac{1}{4}$ $\frac{3}{2} \times \frac{2}{3} \times q = \frac{3}{2} \times \frac{1}{4}$ $q = \frac{3}{8}$	$\frac{3}{8}$

What do you notice about all the second numbers?

You can look at as many examples as you require to find a pattern. Here we notice that every time the second number is less than one. Thus, it seems that if the product is less than the first number, then the second number must be less than one.

Now you try some examples for part (II) and see what is true for the second number in each case.

Read the following example very carefully.

'If a given rational number (less than one) is divided by a rational number between 0 and 1, how does the quotient compare in size with the given number?'

Let's try some numerical examples.

Given Number	Number Between 0 and 1	Finding Quotient	Quotient
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4} \div \frac{1}{2}$ $\frac{3}{4} \times \frac{2}{1}$ $\frac{3}{2}$	$1\frac{1}{2}$
$\frac{3}{5}$	$\frac{4}{15}$	$\frac{3}{5} \div \frac{4}{15}$ $\frac{3}{5} \times \frac{15}{4}$ $\frac{9}{4}$	$2\frac{1}{4}$
$\frac{4}{7}$	$\frac{8}{11}$	$\frac{4}{7} \div \frac{8}{11}$ $\frac{4}{7} \times \frac{11}{8}$ $\frac{11}{14}$	$\frac{11}{14}$

How does the quotient compare in size with the given number? In each case we can see the quotient is greater than the given number, when the given number (less than one) is divided by a rational number between 0 and 1.

Remember that all problems are different. But you may sometimes find it useful to use numerical examples when looking for mathematical relationships or patterns.

The most important thing to do when solving problems is to THINK! THINK! THINK!

Read the following instructions and then do the problems.

1. After trying a problem, check your answer with the one given at the end of the Problem Solving section. If it is incorrect, try the problem again to see if you can arrive at the right answer.
2. Do NOT spend more than 15 to 20 minutes of hard thinking on a single problem. After that time, leave it and try the next problem in the section you are working. You are not expected to do every problem. If you find that the problems in a section are very easy and do not require you to do much thinking, leave this section and start the next section. The problems having been divided into four sections with the easier problems first. For you to be doing real problem solving you should be challenged and be required to think about a problem before being able to arrive at an answer.
3. When you have tried all the problems in a section, return to those you missed. You may be surprised at your second try.
4. Work on some of these problems at home. You may have more time to think about them.
5. Have your teacher look at your "Problem Solving" attempts and achievements.

1. Suppose you saw the two number sentences below written on the chalkboard by two students. What correct conclusion could you reach about these products <u>without doing any computations</u>? Why?

$$\frac{24}{23} \times \frac{65}{66} = \frac{1560}{1518} \qquad \frac{65}{66} \times \frac{24}{23} = \frac{1580}{1518}$$

2. Joe saw the two sentences below written on the chalkboard. He said "If both sentences are true, then addition of rational numbers is NOT commutative. Was he correct? Explain.

$$\frac{6}{8} + \frac{6}{12} = \frac{30}{24} \qquad \frac{6}{12} + \frac{6}{8} = \frac{5}{4}$$

3. Simplify the following:

   a) $\frac{7}{35} + \frac{3}{18} + \frac{8}{105}$

   b) $\frac{5}{24} + \frac{7}{30} + \frac{9}{40}$

4. What rational number less than 2 can be added to $\frac{5}{8}$ so that each sum is greater than 1?

5. Show how the distributive property can be used to show that $\frac{1}{2} \times 6\frac{2}{3}$ is $3\frac{1}{3}$.

6. The universe or replacement set for 'n' in the following conditions of equality is the set of whole numbers. Find replacements for 'n' which make each condition true.

   a) $\frac{n}{n} = \frac{n}{8}$ b) $\frac{n}{n} = \frac{16}{16}$ c) $\frac{4}{5} \times \frac{3}{n} = \frac{3}{10}$

Challengers Section 2.

7. A certain rational number is greater than zero. Its reciprocal is larger than the number itself. What must be true about the number?

8. What can be said about the value of reciprocals of numbers which are very close to one?

9. If a given rational number (greater than zero) is multiplied by a rational number between 0 and 1, how does the product compare in size with the given number?

10. If a given rational number (greater than zero) is divided by a rational number between 0 and 1, how does the quotient compare in size with the given number?

Challengers Section 3.

11. Express the sum of $\frac{x}{y} + \frac{p}{q}$ as a fraction.

12. If a, b, and c are different whole numbers, find the values of a, b and c if:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

13. Write the solutions for the following sentences. The universe is the set of rational numbers greater than zero.

a) $\frac{3}{4} - n = \frac{1}{8}$

b) $n + \frac{2}{3} > \frac{5}{6}$

c) $n - \frac{2}{5} > \frac{1}{5}$

d) $2n > 3$

e) $\frac{2}{3} n > \frac{1}{6}$

14. Find the following sums. Then find a relationship between each sum and the numbers involved.

a) $\frac{1}{1\text{x}2} + \frac{1}{2\text{x}3} =$

b) $\frac{1}{1\text{x}2} + \frac{1}{2\text{x}3} + \frac{1}{3\text{x}4} =$

c) $\frac{1}{1\text{x}2} + \frac{1}{2\text{x}3} + \frac{1}{3\text{x}4} + \frac{1}{4\text{x}5} =$

d) $\frac{1}{1\text{x}2} + \frac{1}{2\text{x}3} + \frac{1}{3\text{x}4} + \frac{1}{4\text{x}5} + \frac{1}{5\text{x}6} =$

(cont.)

14. Use this relationship to find the following sums.

e) $\frac{1}{1x2} + \frac{1}{2x3} + \frac{1}{3x4} + \ldots + \frac{1}{49x50} =$

f) $\frac{1}{1x2} + \frac{1}{2x3} + \frac{1}{3x4} + \ldots + \frac{1}{99x100} =$

15. Find the sums of the following fractions and look for a relationship between the sum and the fractions.

eg. $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

a) $\frac{1}{9} + \frac{1}{18} =$ c) $\frac{1}{15} + \frac{1}{30} =$

b) $\frac{1}{12} + \frac{1}{24} =$ d) $\frac{1}{18} + \frac{1}{36} =$

Using the relationship found above, find the following sums. (Do NOT form the L.C.D. and add the fractions, use the relationship.)

e) $\frac{1}{21} + \frac{1}{42} =$ h) $\frac{1}{45} + \frac{1}{90}$

f) $\frac{1}{24} + \frac{1}{48} =$ i) $\frac{1}{48} + \frac{1}{96}$

g) $\frac{1}{30} + \frac{1}{60} =$

Complete the following sentences using the same pattern as above.

j) $\frac{1}{33} + \frac{1}{\square} = \frac{1}{\square}$

k) $\frac{1}{60} + \frac{1}{\square} = \frac{1}{\square}$

l) $\frac{1}{75} + \frac{1}{\square} = \frac{1}{\square}$

m) Write a formula to express the relationship found.

Challengers Section 4.

16. After much experimenting Robert claimed that dividing rational numbers could be done by dividing the numerators and dividing the denominators of the fractions in much the same way as multiplication of rational numbers. Try some numerical examples to determine if Robert was correct. Prove your answer using rational numbers $\frac{a}{b}$ and $\frac{c}{d}$.

17. Demonstrate on a number line the multiplication of the following rational numbers.

a) $\frac{3}{4} \times \frac{8}{9}$ b) $2 \times \frac{4}{3}$ c) $\frac{3}{2} \times \frac{3}{4}$

18. Indicate diagramatically, with regions, how to form the following products.

a) $\frac{2}{3} \times \frac{9}{4}$ b) $\frac{4}{3} \times \frac{3}{4}$ c) $\frac{4}{3} \times \frac{9}{4}$

19. a) Can the distributive property be used to find the product $6\frac{1}{2} \times 2\frac{2}{3}$? If it can, then show how.

b) If you were in the Advanced Group for this topic, prove the distributive property of multiplication over subtraction of rational numbers. (hint: use the definition of subtraction).

20.
$$\begin{aligned}\frac{11}{4} &= 2 + \frac{3}{4} \\ &= 2 + \cfrac{1}{\frac{4}{3}} \\ &= 2 + \cfrac{1}{1\frac{1}{3}} \\ &= 2 + \cfrac{1}{1+\frac{1}{3}}\end{aligned}$$

$2 + \cfrac{1}{1 + \frac{1}{3}}$ is known as a <u>continued</u> <u>fraction</u> for $\frac{11}{4}$.

It is represented by the symbols (2; 1, 3)

Another example of a continued fraction is:

$$\begin{aligned}\frac{19}{47} &= 0 + \cfrac{1}{\frac{47}{19}} \\ &= 0 + \cfrac{1}{2\frac{9}{19}} \\ &= 0 + \cfrac{1}{2 + \frac{9}{19}} \\ &= 0 + \cfrac{1}{2 + \cfrac{1}{\frac{19}{9}}} \\ &= 0 + \cfrac{1}{2 + \cfrac{1}{2\frac{1}{9}}} \quad = \quad 0 + \cfrac{1}{2 + \cfrac{1}{2+\frac{1}{9}}}\end{aligned}$$

The continued fraction is represented by (0; 2, 2, 9).

* Each numerator in a continued fraction is <u>one</u>.

a) Compute the continued fraction for $\frac{64}{15}$ and write the symbol for it. 237

b) Which fraction is represented by the continued fraction (0; 1, 2, 3)?

Discussion:

If the product is greater than the first rational number, the second rational number must be greater than one.

Section 1

1. One of the products must be incorrect. According to the commutative property of multiplication for rational numbers, they should have equal products.
2. No, he was not correct.

$$\frac{30}{24} = \frac{5}{4} \quad \text{ie. } \frac{6}{8} + \frac{6}{12} = \frac{6}{12} + \frac{6}{8}$$

Thus the commutative property of addition for rational numbers is supported.

3. a) $\frac{31}{70}$ b) $\frac{2}{3}$
4. $\frac{3}{8} < n < 2$ ie. the set of rational numbers from $\frac{3}{8}$ to 2.
5. $\frac{1}{2} \times 6\frac{2}{3} = \frac{1}{2} \times (6 + \frac{2}{3}) = (\frac{1}{2} \times 6) + (\frac{1}{2} \times \frac{2}{3}) = 3 + \frac{1}{3} = 3\frac{1}{3}$.
6. a) 8 b) any whole number greater than zero c) 8

Section 2

7. The number is less than one.
8. The value of the reciprocals is also very close to one.
9. The product is smaller than the given number.
10. The quotient is greater than the given number.

Section 3

11. $\frac{xq + py}{yq}$ 12. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$
13. a) sol: $\frac{5}{8}$ b) sol: all rational numbers greater than $\frac{1}{6}$.

    c) sol: all rational numbers greater than $\frac{3}{5}$.

    d) sol: all rational numbers greater than $\frac{3}{2}$.

    e) sol: all rational numbers greater than $\frac{1}{4}$.

14. a) $\frac{2}{3}$ b) $\frac{3}{4}$ c) $\frac{4}{5}$ d) $\frac{5}{6}$ e) $\frac{49}{50}$ f) $\frac{99}{100}$

15. a) $\frac{1}{6}$ b) $\frac{1}{8}$ c) $\frac{1}{10}$ d) $\frac{1}{12}$ e) $\frac{1}{14}$ f) $\frac{1}{16}$ g) $\frac{1}{20}$

h) $\frac{1}{30}$ i) $\frac{1}{32}$ j) $\frac{1}{33} + \frac{1}{66} = \frac{1}{22}$ k) $\frac{1}{60} + \frac{1}{120} = \frac{1}{40}$

l) $\frac{1}{75} + \frac{1}{150} = \frac{1}{50}$ m) $\frac{1}{3n} + \frac{1}{6n} = \frac{1}{2n}$

Section 4

16. Robert was correct.

Usual method

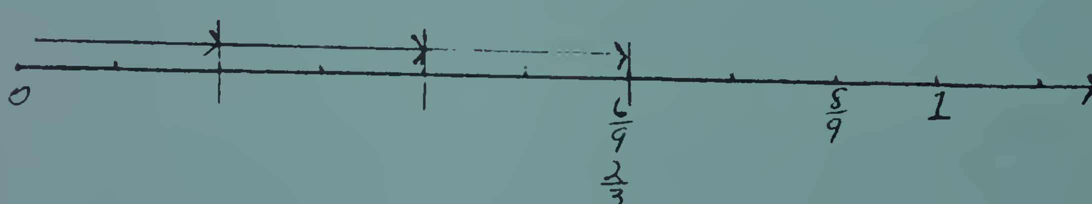
$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$$

Robert's method

$$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d} = \frac{\frac{a}{c}}{\frac{b}{d}} = \frac{a}{c} \times \frac{d}{b} = \frac{a \times d}{c \times b} = \frac{a \times d}{b \times c}$$

Robert's method gives the same answer as the usual method.

17. a)

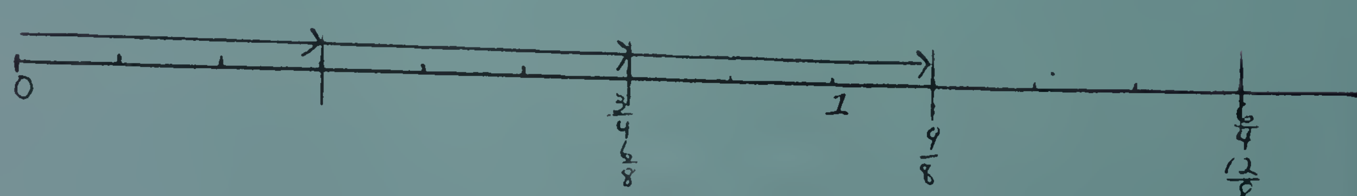


$$\frac{3}{4} \times \frac{8}{9} = \frac{2}{3}$$

b)

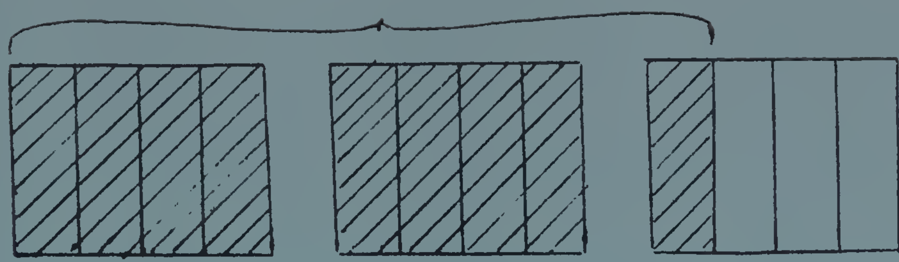
$$2 \times \frac{4}{3} = \frac{8}{3}$$

c)

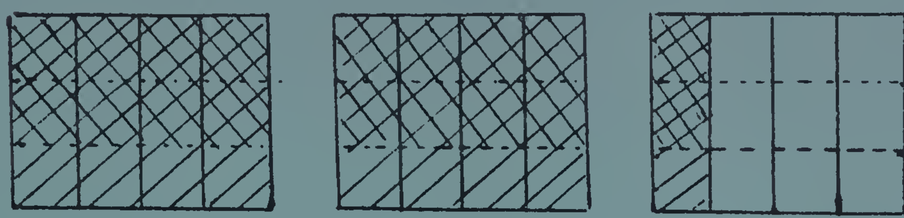


$$\frac{3}{2} \times \frac{3}{4} = \frac{9}{8}$$

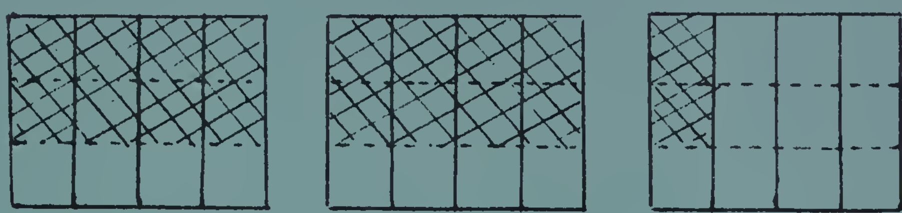
18. a)



$\frac{9}{4}$

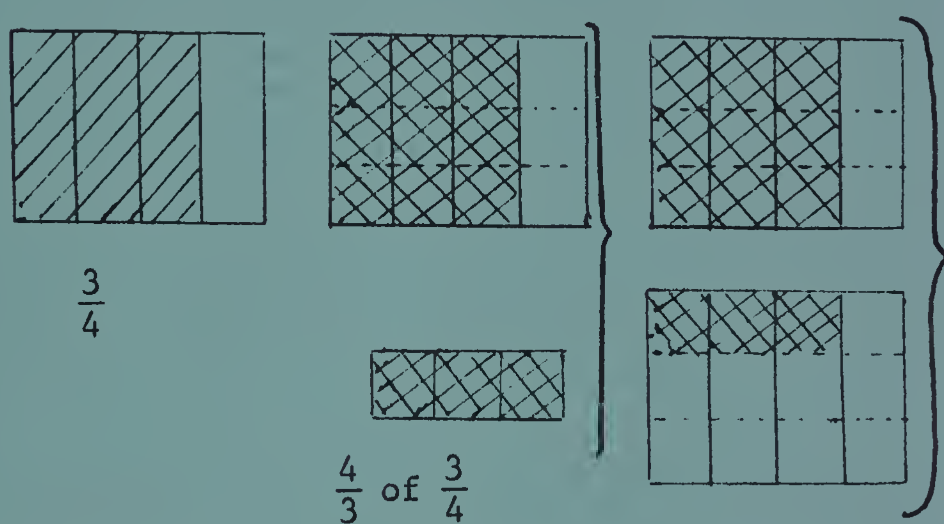


$\frac{2}{3}$ of $\frac{9}{4}$



$\frac{2}{3}$ of $\frac{9}{4} = \frac{18}{12}$

b)

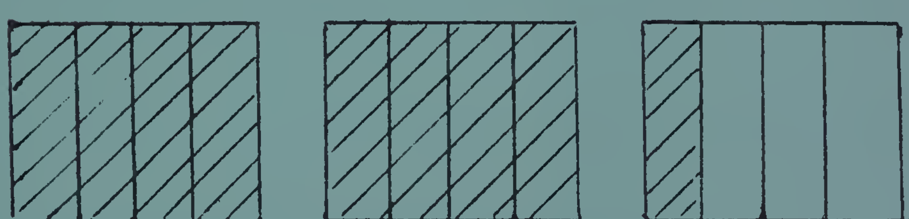


$\frac{3}{4}$

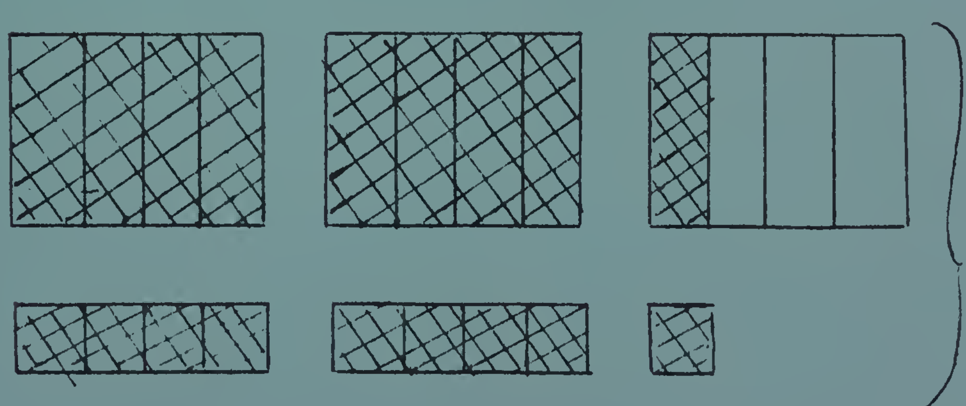
$\frac{4}{3}$ of $\frac{3}{4}$

$\frac{4}{3}$ of $\frac{3}{4} = \frac{12}{12}$

c)

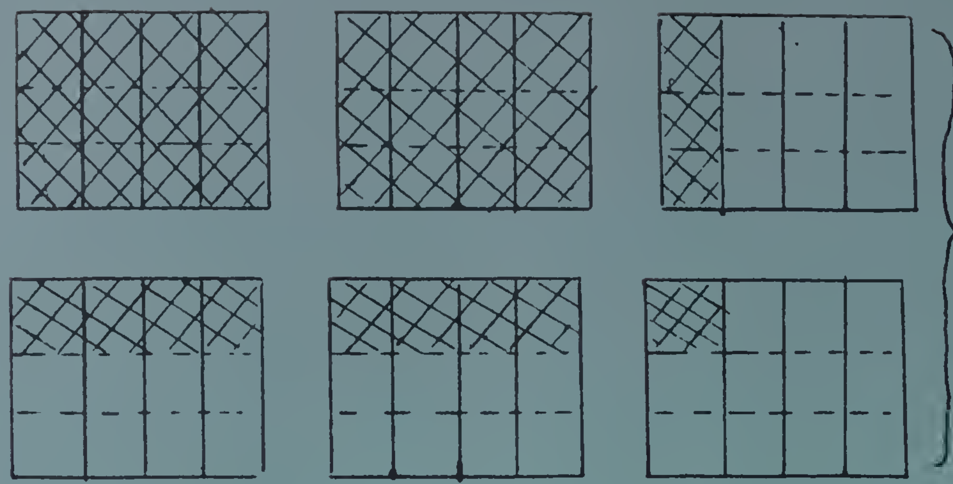


$\frac{9}{4}$



$\frac{4}{3}$ of $\frac{9}{4}$

cont. c)



$\frac{4}{3}$ of $\frac{9}{4} = \frac{36}{12}$

19. a) $6\frac{1}{2} \times 2\frac{2}{3} = 6\frac{1}{2} \times (2 + \frac{2}{3})$

$= (6\frac{1}{2} \times 2) + (6\frac{1}{2} \times \frac{2}{3})$

$= (6 + \frac{1}{2}) \quad \times \quad 2 \quad + (6 \times \frac{1}{2}) \times \frac{2}{3}$

$= (6 \times 2) + (\frac{1}{2} \times 2) + (6 \times \frac{2}{3}) + (\frac{1}{2} \times \frac{2}{3})$

$= \quad 12 \quad + \quad 1 \quad + \quad 4 \quad + \quad \frac{1}{3}$

$= 17\frac{1}{3}$

b) Let $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ be any rational numbers.

Prove: $\frac{a}{b} \times (\frac{c}{d} - \frac{e}{f}) = (\frac{a}{b} \times \frac{c}{d}) - (\frac{a}{b} \times \frac{e}{f})$

Proof:

Statement	Reason	Comments
$\frac{a}{b} \times (\frac{c}{d} - \frac{e}{f})$		Start with left side.
$= \frac{a}{b} \times (\frac{cf - de}{df})$	Definition of subtraction of rational numbers.	
$= \frac{a \times (cf - de)}{b \times (df)}$	Definition of multiplication of rational numbers.	
$= \frac{acf - ade}{bdf}$	Distributive property of whole numbers.	cf and de are whole numbers.
$(\frac{a}{b} \times \frac{c}{d}) - (\frac{a}{b} \times \frac{e}{f})$		Now start with right side and get it to be the same as the fourth line.
$= \frac{ac}{bd} - \frac{ae}{bf}$	Definition of multiplication of rational numbers.	
$= \frac{acbf - aebd}{bdbf}$	Definition of subtraction of rational numbers.	

Statement	Reason	Comments
$= \frac{b(acf - dae)}{bdbf}$	Distributive property of whole numbers.	
$= \frac{\cancel{b}(acf - dae)}{\cancel{b}dbf}$	Reducing the fraction.	
$= \frac{acf - ade}{bdf}$	Commutative property of whole numbers	Order is changed in some terms.
$\therefore \frac{a}{b} \times (\frac{c}{d} - \frac{e}{f}) = (\frac{a}{b} \times \frac{c}{d}) - (\frac{a}{b} \times \frac{e}{f})$		Result is same as left side.

ie. Multiplication is distributive over subtraction of rational numbers.

20. a)

$$\frac{64}{15} = 4 + \frac{4}{15}$$
$$= 4 + \cfrac{1}{\cfrac{15}{4}}$$
$$= 4 + \cfrac{1}{3\frac{3}{4}}$$
$$= 4 + \cfrac{1}{3 + \cfrac{3}{4}}$$
$$= 4 + \cfrac{1}{3 + \cfrac{1}{\cfrac{4}{3}}}$$
$$= 4 + \cfrac{1}{3 + \cfrac{1}{1\frac{1}{3}}}$$
$$= 4 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{3}}}$$

$\therefore$ (4; 3, 1, 3)

b)

$$(0;\ 1,\ 2,\ 3) = 0 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3}}}$$
$$= 0 + \cfrac{1}{1 + \cfrac{1}{2\frac{1}{3}}}$$
$$= 0 + \cfrac{1}{1 + \cfrac{1}{\cfrac{7}{3}}}$$
$$= 0 + \cfrac{1}{1 + \cfrac{3}{7}}$$
$$= 0 + \cfrac{1}{1\frac{3}{7}}$$
$$= 0 + \cfrac{1}{\cfrac{10}{7}}$$
$$= 0 + \frac{7}{10}$$

$\therefore$ the fraction is $\frac{7}{10}$.

Hardisty Junior High School
10534 - 62 Street
Edmonton 80, Alberta

Field Trip - Tues. March 3, 1970

Imperial Oil - Financial Accounting Dept.

I. Introduction:

(1) introduction of students and Imperial Oil personnel.
(2) a check with the students purpose of the field trip (to relate mathematics to the business world)
(3) an explanation of the Imperial Oil Company with particular emphasis on the financial accounting department of the Comptroller's division.

II. Overall Office:

A. "Awe" effect - an explanation of the workings of the general office.

(1) Personnel
(2) Hardware
(3) Use of the machines
(4) Everyone must know their job
(5) Office has a routine:- work - coffee breaks - lunch, etc.
(6) General plan of an office:- office work flow
(7) Efficiency of an office:- lack of errors
- organization of work
- quality of work

III. Particular workings of the office:

Incoming Mail
↓
Comptometer (for checking)

A. Producing Operations B. Marketing C. Logistics D. Freight
↓
Receiving Reports ↔ Invoices
↓
Coders
↓
Voucher Placement
↓
Credits and Debits Balance
↓
Typists for cheques
↓
Audit Department
↓
Approval
↓
Mailing

IV. Look at a Particular Phase - Casual Payroll:

A. Explanation of Casual Help

(1) benefits
(2) hiring
(3) manual processing

B. Gross Pay .... Net Pay

(1) TD1 form
(2) hours of work
(3) vaction pay
(4) Income tax
(5) Canada pension
(6) Unemployment insurance
(7) checking of deductions

V. Summary:

Look at **the** purpose of the visit - was it achieved?

:::::::::::::::::::::

CHAPTER REVIEW

KEY IDEAS

Section 1

Fractions are symbols used to compare parts to a whole, to compare sets, regions, lengths or quantities and for measurement.

Section 2

Diagrams may be used to represent fractions. When shaded parts of regions are used to represent a fraction with numerator greater than denominator, the unit region used should be shown.

Sections 3 and 4

Several different fractions may be represented by the same diagram. Such fractions are called equivalent fractions.

Section 5

A basic fraction is one in which the numerator and denominator have no common factor (other than one).

A fraction equivalent to a basic fraction is obtained by multiplying numerator and denominator of the basic fraction by the same natural number.

A fraction is "reduced" to its equivalent basic fraction by dividing numerator and denominator of the fraction by their greatest common factor.

Section 6

Whole numbers can be represented by fractions. Their basic fractions have denominators of one.

Section 7

With each infinite set of equivalent fractions there is associated exactly one rational number. The basic fraction in the set is usually used to name the rational number.

Each rational number is associated with exactly one point on the numberline.

Equivalent fractions name the same rational number.

Section 8

Whether the fractions are equivalent can be tested either

(1) by reducing them to their basic fractions or

(2) by finding fractions with common denominators and equivalent to the given fractions.

Section 9

Rational numbers can be arranged in order of size by naming them with fractions having common denominators. The numerators are then used to order the fractions.

Section 10

The set of rational numbers is dense because between any two rational numbers there are infinitely many other rational numbers.

To find rational numbers between two rational numbers, the given rational numbers are named by fractions having suitable common denominators.

Section 11

Rational numbers and fractions can be used to answer questions in everyday life.

Summary

Section 1

Addition and subtraction of rational numbers named by fractions is usually done by obtaining equivalent fractions with least common denominators and then adding or subtracting the numerators. Answers are given as basic fractions.

Section 2

A mixed numeral contains a whole number numeral and a fraction.

Each mixed numeral has a corresponding fraction

$4\frac{3}{8} = (4 \times 8) + 3$ eighths

$= 35$ eighths

$= \frac{35}{8}$

AND

Each fraction with numerator greater than denominator has a corresponding mixed numeral.

$\frac{35}{8} = 8\overline{)35} = 4\frac{3}{8}$

Section 3

In addition (or subtraction) of rational numbers named by mixed numerals, the whole numbers are added (or subtracted) and the numbers named by the fractions are added (or subtracted).

Section 4

A solution for a condition for equality is a number, which when put in place of the variable, makes the condition a true statement.

Conditions for equality involving addition (or subtraction) of rational numbers are solved by finding (related conditions with the variable alone on one side.

Section 5

Addition and subtraction of rational numbers are used to answer questions about every day situations. Most examples are done by

(1) finding the relationship between what is asked and what is given.

(2) obtaining the relationship in mathematical form and doing the indicated mathematical work to get a mathematical answer.

(3) Using the mathematical answer to answer the original question.

Section 6

$$\frac{a}{b} \text{ of } \frac{c}{d} = \frac{a \times c}{b \times d}$$

Section 7

Multiplication of rational numbers is done as follows:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

When mixed numerals or whole numbers are involved, their corresponding fractions are used. Answers are given as basic fractions. Reduction to basic fractions is done by dividing numerator and denominator by their common factors (cancelling).

Section 8

The reciprocal of a number is another number whose product with the given number is 1.

The reciprocal of a non-zero rational number is obtained by interchanging the numerator and denominator of the fraction for the rational number.

Zero has no reciprocal.

Section 9

To divide by a non-zero rational number, you multiply by its reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

It is not possible to divide by 0.

Section 10

Conditions for equality involving multiplication of rational numbers can often be solved by finding related conditions involving division of rational numbers with the variable alone on one side.

Also, in conditions for equality in which the variable is multiplied by a number, the variable can be obtained alone on one side by multiplying both sides by the reciprocal of the number by which the variable is multiplied.

Section 11

To answer questions from everyday situations by mathematics we usually find the relationship between what is asked for and what is given. The mathematical form of this relationship is called the mathematical model for the situation. The mathematical model is often a condition for equality. The solution for this condition for equality is used to answer the original question.

Section 12

The rational numbers are closed for addition and multiplication. Addition and multiplication of rational numbers are commutative and associative.

0 and 1 are the identities for addition and multiplication respectively of rational numbers.

Multiplication of rational numbers is distributive over addition. Non-zero rational numbers have reciprocals. This is a property of rational numbers which whole numbers do not have.

CHAPTER REVIEW

KEY IDEAS

Section 1

A number is a power of ten if and only if it can be written as ten to some exponent, or one over ten to some exponent.

$10^0 = 1$

Every digit in a numeral has a place-value.

A numeral can be written in expanded notation by forming the sum of each digit times its place value.

Section 2

A rational number can be expressed by any of three different numerals; a mixed numeral, a decimal numeral, or a fraction. Then every fraction names a rational number which can also be named by a decimal, and vice versa.

Section 3

Addition and subtraction of decimal numbers can be justified by performing the operation with the numerals in expanded notation.

Section 4

Solving conditions with addition and subtraction of decimals is much the same as solving similar conditions with whole numbers.

Section 5

Decimal form is often the most convenient way of expressing rational numbers because they can be added, subtracted and compared more readily than their equivalent fractions.

Section 6

Numbers named by decimal numerals are usually approximate numbers. Decimal numerals can be rounded to however accurate the result needs to be. Rounding is usually done by giving the closest accurate decimal to the given number. A special case occurs when the digit 5 follows the required decimal place.

Section 7, 8

Multiplication and Division of decimal numbers can be justified by performing the operation with the numerals in expanded notation.

Section 8

When dividing rational numbers, it is often necessary to carry the division past the accuracy required and then round off the results.

KEY IDEAS

Section 1

Ratios are numbers which can be used to compare quantities.

A ratio can be written as a fraction $(\frac{4}{3})$, as an implied division $(4 \div 3)$, or with a colon (4:3).

There are equivalent ratios and basic ratios just as there are equivalent fractions and basic fractions.

When two quantities of the same kind are compared using ratios, the same units are used for both quantities.

Section 2

A mathematical statement of equality between two ratios is called a proportion.

A proportion is a true proportion if it has equal cross-products.

Section 3

A proportion containing a variable can be solved by using the cross-product property of proportions. Setting the cross-products equal to each other gives a condition for equality which can be solved in the usual way.

Section 4

Two quantities are proportional (directly proportional) if multiplying one quantity by a factor causes a corresponding use in the other quantity.

Section 5

Proportions can be used to answer questions about everyday situations.

Section 6

Proportions can be used to compare ratios by changing both ratios to have common denominators. Often the common denominator used is 1.

Section 7 & 8

Ratios are easy to compare when written as percentages.

Every ratio can be changed into a percentage by using the rules for fractions to change the denominator to 100; the numerator is then the equivalent percent.

Every percentage can be written as a ratio by putting over 100 and removing the % sign.

Every percentage can be changed to a decimal fraction by removing the % sign and moving the decimal point two places left (dividing by 100).

Every decimal fraction can be changed into a percentage by moving the decimal two places right (Multiplying by 100) and adding a % sign.

Section 9

The concept of percent can be used to answer questions about everyday situations.

Section 10

The regular price of an item multiplied by the discount rate yields the discount.

The regular price minus the discount gives the sale price.

Section 11

The amount of interest payable on a loan is equal to the amount of principle times the interest rate times the number of time intervals for which the money is borrowed.

APPENDIX IV

REVISED MINUTES OF THREE MEETINGS BETWEEN THE EXPERIMENTERS AND TEACHERS

Points raised at a meeting at Hardisty Junior High School on March 6, 1970 concerning the school's mathematics individualization project.

1. Disadvantages of the materials

   - paper work is excessive (tripled)
   - errors found in materials
   - A-group post test II; too many tests. A decision was made to check the unachieved objectives orally for this group, and indicate this on the record sheet.

2. Advantages of materials

   - problems with slow learners are more obvious
   - identification of slow learners is easier
   - teachers have more time to spend with these groups

3. Methods of helping students

   - taking group of slow learners into another room and leaving bulk of class to work alone
   - assigning student helpers to slow learners

4. Another teacher could well be used as a rover; helping out wherever possible.

5. The teacher's role is much different

   - more of a guide than a disciplinarian
   - more team planning, a different sort of individual planning
   - much busier in classrooms than before
   - more attention can be given to slow learners
   - more need for moral support from other teachers

6. Student groups have emerged

   - some need total guidance; 'mother-hen'
   - some are semi independent and need supervision
   - some are completely independent

7. Students want to get into higher groups.

8. Students make good use of student helpers.

9. Problems are being encountered with slower group
   - materials do not provide for them effectively
   - attitude is dropping due to too much reading and a tendency towards disorganization. They experience little success.
   - Suggestion: A more manipulative approach.

- many 'cheated' themselves by not making use of activities and exercises following the first 'check-exercise.' (This is expected to be reduced during Topic 2.)
- this group is not doing less work under the new approach
- they must unlearn six years of training
- this setting provides for easy recognition of these students

10. Modified students

- need different material

11. All students (especially BASIC) need some time to 'goof-off', enter discussions with the teacher, joke, ...
12. Recognition of slow learners and significant individual differences is easier than before.

13. Program relys heavily on compatibility of group of teachers

- teachers must be committed to program
- teachers must be able to work effectively as a team
- a group of less than five could still be effective (a plan for only three teachers is proposed for next year)
- even one teacher might work effectively

14. Teachers must cooperate

- to share the work load
- a single teacher would have five times the work
- need moral support from other team members

15. BASIC and INTERMEDIATE groups need enrichment as well as ADVANCED

16. Is there a tendency towards loosing identity of class

17. Students must change their expectations of the teacher

18. Are there some sections that can be covered better verbally? Should this be done?

19. Slow classes will start Monday (March 9)
- one teacher for each class at all times.

20. The main problem of the whole program seems to be a poor or absent method for catering to the slow learner.

Points raised at the second meeting (March 20, 1970) at Hardisty Junior High School concerning their mathematics individualization project.

1. Discussion of the Phase 2 ADVANCED level

- designed to go more deeply into the ideas of Phase 1 with emphasis on justification of methods and generalizing to other frames of reference
- designed to spread wider and relate other connected topics to the ideas presented in Phase 1.
- designed to pose a challenge to the abler students
- designed to be a more rigorous (more 'mathematical') approach to the ideas of Phase 1.

- not all ADVANCED students found it attractive
- difficulty with terminology and semantics
- difficulty in verbalizing reasons and explanations
- dissapointment at being given more objectives to complete

- some students may not try for ADVANCED group next time

- preconceived attitudes about learning of mathematics may be causing this problem
- attitudes may change dynamically as the students interact with the material, or
- a selling job may have to be done by the teachers - " That is what mathematics is all about."

2. Two main types of students enter the ADVANCED Level.
- those who are mathematically oriented
- those who like to achieve

- The first group was the one expected; it is the second group that is causing concern.

- The ADVANCED material could be made more attractive to this group by
    - a selling job by the teachers
    - inclusion of 'carrots' to make the ADVANCED work more fun.

- Currently incentives for ADVANCED work include:
    - recognition and higher marks
    - extended privleges (going to the reference library and Enrichment center)
    - exposure to more challenging mathematics (?)

- Other considerations
    - students may be able to cope with ADVANCED level ideas better after exposure to the tone of these sections from previous topics.

- the skills required may take a long time to develop (analysis, synthesis, generalization, etc.)
- these students should understand that they are only cheating themselves if they fail to be exposed to these different and higher-level ideas.

3. <u>Discussion of the Enrichment section</u>

- Enrichment material is similar in level and tone to the ADVANCED materials.

- Currently only the ADVANCED students are making good use of the Enrichment materials.

- Materials of a more recreational nature are needed for the slower students.
    - Perhaps concrete materials for developing BASIC concepts (especially for teacher-taught BASIC class)

4. <u>Discussion of planning and Preperation by the teacher</u>

- a very different type of planning is needed
    - not less individual planning, but individual planning of a different sort.
    - more planning for individual students
    - cooperative (team) planning

- Need a cooperative group of teachers (one 'weak-link' would be disruptive to the whole program)

- Teachers must be experienced and have a good grasp of teaching in general to work effectively on this team. (Allowing student freedom and at the same time getting them to accept the corresponding responsibility is very difficult, but all members of the team must be able to accomplish this.)

5. <u>Discussion of Problem Solving versus Applications sections</u>

- Confusion has arisen as to the direction and content of these sections.

- The application sections contain <u>routine</u> problems. Students are encouraged to employ any method (brainstorming, analogy, etc.) to solve them. Problem types vary greatly but are aimed at reinforcing or introducing necessary skills.

- Is it necessary to establish good work habits for problem solving? Will students get 'in a rut' because of routine approaches?

- It was decided to rename the Applications section to Applications (Routine Problem Solving) and the problem solving section to Challengers (Non-Routine Problem Solving). The teachers will further clarify the distinction to the students.

- The teachers' work load is such that they will be unable to evaluate each student's progress on the Challengers section. Therefore a new column on the record page will be introduced to distinguish whether the teacher has marked the solution to a problem or not.

6. It was generally agreed that although the materials do not effectively meet many individual differences, they do make these differences more apparent to the teacher as well as create a situation in which the teacher can effectively deal with some of these differences.

Points raised at the third (and final) meeting of April 28, 1970, concerning the Hardisty Project.

1. <u>There is still a problem with the Challengers</u>

   - they are too hard for the bottom half of the students
   - these students should perhaps be allowed to play a game instead (gin rummy was mentioned)
   - the non-routine approach to problem solving is premature at this grade level

   - neatness and orderliness must be instilled for good problem-solving habits to be formed

   - the teachers have reverted to a 'five-step' approach rather than the 'three-step' approach used in the materials.

   - with a different build-up in earlier grades, this type of problem might be very worthwhile; but for these students at this time, the approach used is not working satisfactorily.

2. <u>Too many clerical duties</u>

   - the children are given too many tests; they are difficult to keep organized
   - the tests take too long to mark
     - multiple-choice questions should be increased

   - keeping records on all students is becoming a burden

3. Some of the problems in the materials are too hard for the students. The numbers used in some conditions are unnecessarily large. Some of the problems have 'sneaky' twists in them.

APPENDIX V

THE STUDENT QUESTIONNAIRE AND RESULTS

The following questionnaire was administered to all (about 270) students in the experimental classes at Hardisty school near the end of the project. All results are reported in percentages. Under TOTAL is the percentage of all students choosing that answer. Under BASIC is the percentage of students with that choice who remained in the Basic group for the whole experiment. Under INTER is the percentage of students who were in the Intermediate group at least once but were never in the Advanced group. Under ADVCD, the percentage of students with that choice who were in the Advanced group at least once is reported.

STUDENT QUESTIONNAIRE

1. During Topic 1, I was in the _____ group.

	BASIC	INTER	ADVCD	TOTAL
a) Basic	100.0	42.9	1.2	39.6
b) Intermediate	0.0	57.1	28.2	37.7
c) Advanced	0.0	0.0	70.6	22.6
d) (not used)	0.0	0.0	0.0	0.0

2. During Topic 2, I was in the _____ group.

	BASIC	INTER	ADVCD	TOTAL
a) Basic	100.0	17.3	2.4	27.3
b) Intermediate	0.0	82.7	38.1	53.8
c) Advanced	0.0	0.0	59.5	18.9
d) (not used)	0.0	0.0	0.0	0.0

3. During Topic 3, I was in the _____ group.

	BASIC	INTER	ADVCD	TOTAL
a) Basic	100.0	36.1	1.4	38.1
b) Intermediate	0.0	63.9	36.2	43.6
c) Advanced	0.0	0.0	62.3	18.2
d) (not used)	0.0	0.0	0.0	0.0

The next 14 questions are about the usefulness of certain aspects of the material in helping you to learn the mathematics.

4. The Introduction to each Topic.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	23.4	13.5	8.2	13.6
b) Of no use	12.8	8.3	20.0	12.8
c) Of some use	48.9	56.4	51.8	53.6
d) Very useful	14.9	21.8	20.0	20.0

5. The list of objectives for each section.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	6.4	6.0	4.7	5.7
b) Of no use	14.9	12.0	3.5	9.8
c) Of some use	42.6	47.4	35.3	42.6
d) Very useful	36.2	34.6	56.5	41.9

6. The development in each section.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	12.8	12.8	13.1	12.9
b) Of no use	8.5	10.5	6.0	8.7
c) Of some use	48.9	40.6	33.3	39.8
d) Very useful	29.8	36.1	47.6	38.6

7. The questions asked IN THE DEVELOPMENT.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	19.1	9.0	4.8	9.5
b) Of no use	12.8	12.8	16.7	14.0
c) Of some use	46.8	54.9	61.9	55.7
d) Very useful	21.3	23.3	16.7	20.8

8. Answers for the development section given at the bottom of the page.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	8.5	3.8	2.4	4.2
b) Of no use	17.0	14.5	14.3	14.9
c) Of some use	51.1	52.7	47.6	50.8
d) Very useful	23.4	29.0	35.7	30.2

9. The check-exercises.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	2.1	0.8	0.0	0.8
b) Of no use	0.0	3.8	2.4	2.6
c) Of some use	17.0	18.8	14.1	17.0
d) Very useful	80.9	76.7	83.5	79.6

10. The activity exercises (after the first check-exercise) in each section.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	6.4	4.5	0.0	3.4
b) Of no use	19.1	16.5	11.9	15.5
c) Of some use	53.2	51.9	53.6	52.7
d) Very useful	21.3	27.1	34.5	28.4

11. The list of key ideas at the end of each topic.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	10.6	6.8	1.2	5.7
b) Of no use	19.1	24.2	19.0	21.7
c) Of some use	48.9	50.8	48.8	49.8
d) Very useful	21.3	18.2	31.0	22.8

12. The vocabulary list.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	11.1	4.5	2.4	5.0
b) Of no use	37.8	32.3	25.3	31.0
c) Of some use	51.1	50.4	53.0	51.3
d) Very useful	0.0	12.8	19.3	12.6

13. The review exercises at the end of each topic.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	6.4	2.3	0.0	2.3
b) Of no use	17.0	7.6	7.2	9.2
c) Of some use	42.6	34.8	18.1	30.9
d) Very useful	34.0	55.3	74.7	57.6

14. Having the solutions at the end of each topic.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	4.3	1.5	2.4	2.3
b) Of no use	14.9	6.8	2.4	6.8
c) Of some use	27.7	26.5	24.7	26.1
d) Very useful	53.2	65.2	70.6	64.8

15. The record page.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	14.9	3.1	2.4	4.9
b) Of no use	14.9	16.0	12.9	14.8
c) Of some use	36.2	47.3	52.9	47.1
d) Very useful	34.0	33.6	31.8	33.1

16. The flowchart.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	12.8	3.0	2.4	4.5
b) Of no use	27.7	32.3	38.8	33.6
c) Of some use	36.2	43.6	37.6	40.4
d) Very useful	23.4	21.1	21.2	21.5

17. Using different colored pages.

	BASIC	INTER	ADVCD	TOTAL
a) Confusing	10.9	6.8	2.4	6.1
b) Of no use	23.9	12.0	10.6	13.6
c) Of some use	28.3	44.4	47.1	42.4
d) Very useful	37.0	36.8	40.0	37.9

For the following questions, choose one of the provided answers to best describe what you did, or your opinions.

18. How did you study each section?

	BASIC	INTER	ADVCD	TOTAL
a) I read the objectives first. If I felt I already understood the ideas, I skipped to the check-exercise; otherwise I read the development.	36.2	33.8	48.2	38.9
b) I read the development first, then I worked the example in the objective. If I understood the example I then tried the check-exercise; otherwise I read the development again.	21.3	22.6	18.8	21.1
c) I worked the check-exercise first. If I was unable to do it I went back to the development.	25.5	30.1	20.0	26.0
d) I did the sections in different ways, depending on how my work had gone with previous sections.	17.0	13.5	12.9	14.0

19. The Phase 1 booklets were made to teach you new ideas. How do they compare with having the ideas taught by a teacher?

	BASIC	INTER	ADVCD	TOTAL
a) I would rather have the teacher teach the ideas.	29.8	29.3	9.4	23.0
b) They are about the same.	19.1	13.5	18.8	16.2
c) It depends on the ideas, sometimes the booklets are better than a teacher.	25.5	38.3	45.9	38.5
d) The booklets are usually better for learning new ideas. ..	25.5	18.8	25.9	22.3

20. How did you like learning these new ideas from the booklets?

	BASIC	INTER	ADVCD	TOTAL
a) I didn't like it at all.	19.1	12.0	8.2	12.1
b) I didn't dislike it any more than any other method.	23.4	10.5	8.2	12.1
c) It was okay but I would not want to do it all the time. .	25.5	35.3	34.1	33.2
d) I liked it very much. ...	31.9	42.1	49.4	42.6

21. Does it take any longer to learn mathematics from the booklets?

	BASIC	INTER	ADVCD	TOTAL
a) It takes much longer than when taught by a teacher.	25.5	15.0	5.9	14.0
b) It takes a little longer.	17.0	13.5	14.1	14.3
c) It depends on the ideas, sometimes the booklets are faster.	31.9	48.1	51.8	46.4
d) The booklets usually teach faster than a teacher.	25.5	23.3	28.2	25.3

22. How do you feel about your learning from the booklets?

	BASIC	INTER	ADVCD	TOTAL
a) I learn ideas better from the teacher.	32.6	23.3	15.3	22.3
b) I learn ideas about the same by either method.	8.7	18.0	20.0	17.0
c) It depends upon the ideas. For some ideas, I learn better from the booklets than I would from a teacher.	37.0	39.8	43.5	40.5
d) I learn most ideas better from the booklets.	21.7	18.8	21.2	20.1

23. How often did you attend the teacher-taught class?

	BASIC	INTER	ADVCD	TOTAL
a) Never.	6.4	10.5	15.3	11.3
b) Seldom.	48.9	47.4	67.1	54.0
c) Often.	23.4	27.1	11.8	21.5
d) Usually.	21.3	15.0	5.9	13.2

24. Why did you attend the teacher-taught class?

	BASIC	INTER	ADVCD	TOTAL
a) I didn't attend.	10.6	11.3	20.0	14.0
b) I was told to attend. ...	10.6	5.3	4.7	6.0
c) It is easier to learn from a teacher.	25.5	18.0	9.4	16.6
d) I needed help with certain ideas.	53.2	65.4	65.9	63.4

25. How did the amount of work you did during mathematics classes compare with the amount in a regular classroom with a teacher?

	BASIC	INTER	ADVCD	TOTAL
a) I did less work with the booklets.	34.0	19.5	25.9	24.2
b) About the same.	27.7	41.4	34.1	36.6
c) I did a little more work while using the booklets.	27.7	27.1	23.5	26.0
d) I worked much harder on math while using the booklets. ..	10.6	12.0	16.5	13.2

26. Did the booklets affect the amount of homework that you did?

	BASIC	INTER	ADVCD	TOTAL
a) I did less homework with the booklets.	27.7	41.4	51.8	42.3
b) I did about the same. ...	34.0	29.3	28.2	29.8
c) I did a little more homework because of the booklets. ...	19.1	20.3	11.8	17.4
d) I did much more homework because of the booklets.	19.1	9.0	8.2	10.6

27. The Phase 2 booklets acted as a review of the topic for the Basic and Intermediate groups, and as a presentation of new ideas for the Advanced group. How worthwhile do you feel the time spent on Phase 2 was?

	BASIC	INTER	ADVCD	TOTAL
a) It was a waste of time for most people.	4.3	7.6	5.9	6.4
b) It was worthwhile for some people, but not for me.	17.0	12.9	15.3	14.4
c) It was useful, but used up too much time.	44.7	36.4	30.6	36.0
d) It was very worthwhile for me.	34.0	43.2	48.2	43.2

28. If you could be in one group all of the time, which would it be?

	BASIC	INTER	ADVCD	TOTAL
a) Basic	25.5	3.0	0.0	6.0
b) Intermediate	46.8	76.7	34.1	57.7
c) Advanced	14.9	6.8	35.3	17.4
d) It would depend on the ideas to be learned.	12.8	13.5	30.6	18.9

29. How do you feel about the Advanced group?

	BASIC	INTER	ADVCD	TOTAL
a) Only people who work very hard get in the Advanced group. .	25.5	25.0	20.0	23.5
b) Only people who really like mathematics and are naturally good at it get into the Advanced group.	42.6	31.1	23.5	30.7
c) Both sets of people a) and b) get into the Advanced group. .	25.5	42.4	55.3	43.6
d) Getting into the Advanced group is more by luck than anything else.	6.4	1.5	1.2	2.3

30. Would (did) you like being in the Advanced group?

	BASIC	INTER	ADVCD	TOTAL
a) No, the ideas are too hard.	32.6	29.3	17.6	26.1
b) No, I would have to work harder than I wanted to.	16.3	27.1	14.1	21.1
c) Yes, the Advanced group gets higher marks.	25.6	21.8	21.2	22.2
d) Yes, the Advanced ideas are interesting.	25.6	21.8	47.1	30.7

31. How did most people feel about your group?

	BASIC	INTER	ADVCD	TOTAL
a) They thought we were pretty 'dumb' at math.	39.1	9.0	3.5	12.5
b) They thought we were 'show-offs'.	6.5	3.0	2.4	3.4
c) No one really cared what group I was in.	32.6	58.6	44.7	49.6
d) None of the above are right.	21.7	29.3	49.4	34.5

32. The Challengers were ...

	BASIC	INTER	ADVCD	TOTAL
a) too hard for grade seven.	21.7	9.8	10.7	12.2
b) very hard, more practice with easier problems would have helped.	17.4	27.8	33.3	27.8
c) very hard, the teachers should have helped us more.	15.2	15.8	16.7	16.0
d) not too hard.	45.7	46.6	39.3	44.1

33. Doing the Challengers section ...

	BASIC	INTER	ADVCD	TOTAL
a) was a complete waste of time.	17.4	22.1	33.3	24.9
b) would have been more useful in grade 8 or 9.	32.6	28.2	16.7	25.3
c) would have been better if we'd had practice in earlier grades.	21.7	27.5	25.0	25.7
d) was very useful, we need to be able to solve problems of this type.	28.3	22.1	25.0	24.1

34. The enrichment in the Enrichment room was ...

	BASIC	INTER	ADVCD	TOTAL
a) too hard for me.	0.0	3.1	1.2	1.9
b) not very interesting to me.	18.2	16.2	10.6	14.7
c) very enjoyable.	20.5	22.3	32.9	25.5
d) I didn't try it.	61.4	58.5	55.3	57.9

35. The review section at the end of Phase 1 was ...

	BASIC	INTER	ADVCD	TOTAL
a) of little use.	17.0	9.0	7.1	9.8
b) of some use, but not really necessary.	25.5	14.3	11.9	15.5
c) could have been more useful if done some other way. (Perhaps a teacher conducting the lessons.)	31.9	30.8	11.9	25.0
d) very useful for tying the ideas of the topic together.	25.5	45.9	69.0	49.6

36. I prepared for the first test in each topic by ...

	BASIC	INTER	ADVCD	TOTAL
a) going through the review section.	27.7	23.5	20.2	23.2
b) doing the review and studying the objectives for each section.	36.2	39.4	31.0	36.1
c) doing the review and checking back to the development when necessary.	29.8	28.0	31.0	29.3
d) special preperation was not necessary.	6.4	9.1	17.9	11.4

37. On the whole, the tests were ...

	BASIC	INTER	ADVCD	TOTAL
a) too hard, you had to be lucky to get a good mark.	14.9	3.0	2.4	5.0
b) too hard, you had to get the question exactly right to get a mark for it.	42.6	23.5	3.6	20.6
c) not too hard, you knew what all of the questions were going to be like.	40.4	62.1	73.5	61.8
d) not too hard, the ideas were covered so many times they were easy to remember.	2.1	11.4	20.5	12.6

38. Was there any effect between this style of learning and your other regular classes?

	BASIC	INTER	ADVCD	TOTAL
a) Yes, it was nice to have a change.	43.2	48.9	60.7	51.7
b) Yes, but it was hard to change from one style of learning to the other.	29.5	18.3	11.9	18.1
c) Yes, but it was a very small effect.	15.9	17.6	9.5	14.7
d) No, there was no effect.	11.4	15.3	17.9	15.4

39. Would you like to see this type of learning tried in some of your other classes?

	BASIC	INTER	ADVCD	TOTAL
a) Yes, in all of them.	17.0	16.5	17.6	17.0
b) Yes, but only in some certain ones.	36.2	51.1	50.6	48.3
c) No, this probably only works well with mathematics.	25.5	20.3	28.2	23.8
d) No, I don't like this method at all.	21.3	12.0	3.5	10.9

40. On the whole, do you feel that you have learned more about mathematics this year than in other years?

	BASIC	INTER	ADVCD	TOTAL
a) Yes, and it is because of the booklets.	40.0	39.8	48.2	42.6
b) Yes, but I don't think the booklets made any difference. ...	20.0	30.1	35.3	30.0
c) No, I didn't like the booklets.	17.8	15.0	9.4	13.7
d) No, but I don't think the booklets made any difference. ...	22.2	15.0	7.1	13.7

SUBJECTIVE ANSWER SHEET

Use this sheet to answer the very last question only.

COMMENTS:

I found the new booklet method very easy to learn by, but the coverage of knowledge in the old method was greater. The old method promoted a better memory of the work done. a great deal less was learned this year as compared to most. To make the new method efficient, more exercises and lessons should be compulsory and done in a shorter time. At the present, there is a sparing amount of work done in the average math period.

Two things you liked MOST

1. I liked learning for myself which avoided confusion in some cases. 2. I liked advancing in my own speed, which prevents setbacks from students in the old method.

Two things you liked LEAST

1. The total consumption of learning was less.

2. The white pages were more confusing than assisting, and to achieve a misunderstood lesson I had to look at the objective pages and attempt to figure out the method

SUBJECTIVE ANSWER SHEET

Use this sheet to answer the very last question only.

COMMENTS: I would be doing better I think if were doing mathematics like we used to in grade six because in the grade six form I prefer listening from the teacher. I also preferred that kind of work from the textbook instead of this booklet.

Two things you liked MOST

Having quizzes
Having mini lectures

Two things you liked LEAST

Studying
Homework

SUBJECTIVE ANSWER SHEET

Use this sheet to answer the very last question only.

COMMENTS:

I think you should have a little more time instead of having to have 5 lessons done in 2 days.

Two things you liked MOST

Its easyer with the booklets.
I learn new things

Two things you liked LEAST

Some questions are too hard.
Not enough time.

SUBJECTIVE ANSWER SHEET

Use this sheet to answer the very last question only.

COMMENTS: I think math this year was alot better than last year. I hated math last year, but I like it this year. I like the mini lectures very much I like the teachers teaching me. I don't like taking tests, and I liked my teacher. I liked working at my own speed.

Two things you liked MOST

Mini Lectures

And working at our own speed.

Two things you liked LEAST

I don't like learning on my own.